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Conversion from conventional life insurance policies into universal life policies

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CONVERSION
FROM
CONVENTIONAL
LIFE INSURANCE POLICIES
INTO
UNIVERSAL LIFE POLICIES

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CONTENTS

ABSTRACT

This paper considers different aspects of conversion from conventional life insurance policies into universal life policies.

Finding formulas for conventional policies on an annual basis is typically quite straightforward, but this paper analyses and includes more complicated cases also.

Monthly mortalities are also discussed in this paper. This paper introduces a concept "discount factor preserving method" which ensures the compatibility of old and new formulas. Also some other methods are derived to solve other challenges.

The main focus of this paper is conversion. The results, especially when viewed from or analyzed on a monthly basis, are different than those referred to in actuarial literature.

This paper also examines the accounting viewpoint, where calculations are performed at the end of a calendar month and not on full months or years after the effective date of the policy.

This paper suggests that the leadership and management of the insurance undertaking should consider the possibility of changing the technical bases of their products in specific cases in order to create systems that are more cost effective and policies that can be managed more easily.

1 INTRODUCTION

1.1 Purpose of this document

This paper describes conventional model conversion into universal life models. Some approximation models are also considered in this paper.

I have chosen to use the terms

- *conventional model* for the prospective model where the liability is the present value of the future net outgoing cash flows [1]
- *universal life* model for the retrospective model where the liability is calculated as the accumulation of account entries over the years up to the balance sheet date[2]

This paper concentrates on modeling the reserve calculation using universal life models.[3]

1.2 Examples and industry practices

The examples are mostly based on industry practices that I have seen used in Finland and Estonia.

In order to preserve the confidentiality of the undertakings the models described here are generalized models and are not exact examples used by any particular undertaking. For instance the loading structures and the parameter values differ slightly from the practice. I also describe some alternative solutions and do not describe which model some undertakings have used. If I propose the use of some alternative, this proposal should be taken as my personal view only.

[1] I have chosen to use term "conventional model" instead of "traditional model" as it has been called e.g. by Angus S. MacDonald. (See MacDonald p. 980) and Black et al. (See Black et al, p. 113)

[2] The universal life model can sometimes also been called the "recursive method "(See e.g. Gerber, p. 68 and Savolainen). However, recursive methods are used also in other contexts. Using the name "recursive method" is justified especially when the conventional formulas are expressed in universal life formulas.

[3] By "reserve" in this document I mean the policy savings and not "liabilities" because in IFRS the liabilities may differ from the reserve and at least in phase I the savings may be divided into the financial liabilities and insurance liabilities portions. In general actuarial literature, typically the term "reserve" has been used.

1.3 Used notation

The notations used in this document are mainly based on the International Actuarial Notation, published in the Encyclopedia of Actuarial Science.[4] However, in this document I do not always differentiate between discrete and continuous models because sometimes the same formulas may be used for both models. The difference is only specifically cited in situations where there is some relevant difference between the models.

About the notation of ages see the footnote 15.

The terminology and notations used may differ from one reference to another, but when cited in this document, uniform terminology and symbols are used.

[4] See Wolthuis, pp. 927 – 931. See also chapter 5.

2 MOTIVATION FOR THE PAPER

2.1 New requirements for the insurance undertakings

New requirements laid down for the insurance undertakings do not require universal life models. However, using universal life models may help in the process of fulfilling requirements.

IFRS 4 requires an insurance undertaking for example
- to unbundle deposit components and insurance components of some insurance contracts[5]
- to carry out a liability adequacy test[6]
- to perform sensitivity analysis[7]

In the solvency II framework[8] it is likely that an insurance undertaking also
- may develop their own internal models to assess their risks and solvency
- may and sometimes shall have to perform specific tests based on their insurance portfolio

In Principles for the Conduct of Insurance Business IAIS has stated about the disclosure principles that the insurance undertaking
- shall have to inform the policyholder of costs and associated charges[9]

As a result of these requirements insurance undertakings should have more detailed information about their portfolios. This paper does not discuss the simulation of future cash flows but concentrates on modeling the reserve calculation using universal life models.[10]

Some approximation methods can also be used, like those described in the guidelines given by the Ministry of Social and Health Affairs.[11] However, those tools which are based on the average numbers of the accounting period were developed during times when it was not possible to use effective computers to manage large quantities of data. At present more accurate methods can be used.

[5] See IFRS 4, 10 – 12.
[6] Ibid 14 (b), 15 – 19.
[7] Ibid 39 (c)(i) and IFRS 4 Guidance, IG 52 – 54.
[8] See "Solvency II Proposal" art. 34 and subsection 3.
See European Commission, Financial Institutions, point 5, p. 3.
[9] See about this transparency requirement International Association of Insurance Supervisors, 12.3, p. 7.
[10] See about modeling e.g. Koller and IAA.
[11] See Koskinen et al. and Sosiaali- ja terveysministeriö

2.2 Efficiency of the insurance undertaking processes

It is well documented that old insurance undertakings have applied many techniques during their product generations and life cycles. It is also well known that the new ones are universal life -type policies. This means that the products in conventional techniques are often run-off portfolios that will still be in force for many years to come.

It is inevitable that sooner or later, the leadership of insurance undertakings will have to ask themselves whether it is commercially viable or indeed is it even wise to maintain these old techniques because of the financial implications involved of running separate computer systems, improving and increasing their efficiency and then needing and also requiring and employing several interfaces to consolidate the data.

One option is to convert the policies with conventional techniques to policies with universal life techniques. Then, if the software is parameter driven, sometimes the undertaking may be able to manage several products with the same software.

2.3 Lack of knowledge of the conversion

Over the years in discussion with actuaries in insurance undertakings I have realized that people are often not familiar with the conversion formulas. One reason may be that the literature does not directly derive the formulas from the traditionally used commutation numbers. Actuaries want to verify that the formulas work in their products. This paper should assist in the verification process.

The formulas in this paper have been tested in practice by calculating the same numerical input data using both the conventional model and the universal life model.

Typically the formulas are written for a sum insured equal to 1, which entails that the reserve has to be multiplied by the sum insured. However, in practice the formulas are calculated based on the actual sum insured, and this convention is therefore used in this paper.

3 STRUCTURE OF THE DOCUMENT

The paper also develops a step by step approach to the conversion rules:

1) Introducing and noting differences that are essential from the business perspective.
2) Introducing a general annual model without any loadings. In this context annual model means that the calculations are performed annually only at the end of an insurance year.[12]
3) Introducing a general annual expense-loaded model.
4) Highlighting some extensions to annual expense-loaded model.
5) Discussing changes in sums and premiums
6) Discussing issues related to monthly expense-loaded models. In this model the calculations are performed monthly at the end of an insurance month.
7) Looking at issues related to performing calculations at the end of a calendar month.
8) Discussing sum issues related to approximations

Finally this paper will provide a short summary of the results.

[12] In this paper, by "insurance year" I mean the one-year-long time period starting the same month and day as the policy becomes effective. By "insurance month" I mean any one-month-long time period starting the same day as the policy becomes effective. If the month does not have that day, the last day of the month is chosen.

4 DIFFERENCES BETWEEN UNIVERSAL LIFE AND CONVENTIONAL PRODUCTS

4.1 General

There are some essential differences between universal life and conventional products that are relevant for the purpose of this paper:

4.2 Reserve calculation formulas

In conventional policies the reserves can be calculated using a small amount of data. Some information on the policy is needed, like information on the insured, on sum insured, policy period left and future payments.

In universal life policies the reserves are calculated using the information of the previous reserve value and the change of reserve is calculated using for example information on insured and sum insured. All changes to reserves have to be stored, and this requires computer and storage capacity.

4.3 Payment schedule flexibility

In conventional products the reserves are calculated assuming that the payments have been paid exactly on their due dates. When calculating the liabilities, unearned premiums and prepayments have to be taken into account separately. In universal life product the paid payments are not considered to be unearned premiums when they have been allocated to savings, but in some cases premiums.

If the universal life formulas would be used for conventional products, then one option is that payments are added to reserves not when paid but rather on their due dates. This means that unearned premiums and prepayments have to be analyzed separately.

As in pure universal life products the policyholder is not so tightly committed to the payments, so it is common to define separate penalty loadings in case the policyholder does not meet the level of the original payment schedule.

4.4 Investment flexibility

In universal life products it is possible for the policyholder to invest in several investment instruments. In this case the reserve is divided into several accounts and the accounts keep records on the units invested into funds. This type of policy is often called unit-linked policy.[13] In this method, the investment risk is borne by the policyholder rather than the insurance undertaking.

4.5 Calculation of reserves

In the case of universal life products the reserves are calculated at least once a month, although for example the effects of payments and surrender values can be analyzed daily. In the case of conventional products, the calculations are performed when needed, for example for financial statements and customer information.

[13] See e.g. Richards, pp. 1716 – 1724. In the USA this is often called "Variable universal life" (See e.g. Black et al. (See Black et al, p. 127)

5 CONVENTIONAL FORMULAS

5.1 General

In this paper I do not derive the conventional formulas because the derivations are readily available from the literature.[14]

I shall provide proof and substantiate the effectiveness of some simple formulas that I will use towards the end of this chapter.

5.2 Discrete model

In a discrete model mortalities are calculated separately for each age.[15] For each age x a number l_x is estimated. This number represents people alive at age x (it can be assumed e.g. that $l_0 = 10^6$).

Let us define

$$d_x = l_x - l_{x+1}$$
$$q_x = \frac{d_x}{l_x} = 1 - \frac{l_{x+1}}{l_x}$$
$$v = \frac{1}{1+i}$$

where

x	is age
d_x	is number of death (at age x)
q_x	is mortality (at age x)
i	is technical interest rate
v	is discount coefficient

[14] See e.g. Gerber, pp. 119 – 123, Neill pp. 38 – 71, Schmidt, pp. 123 –129 and Pesonen et al., pp. 54 – 78.
[15] Unless otherwise mentioned, for ages I have used the following notifications:

- x	the age at the beginning of the insurance year
- x+t	the age t whole years after x
- x+t+m/12	the age m whole months ($0 \le m < 12$) from x+t (before x+t+1)
- x+t+u	the age time u ($0 \le u < 1$) from x+t

The commutation numbers are as follows:

$$D_x = l_x \cdot v^x$$

$$N_x = \sum_{i=x}^{w'} D_i$$

$$C_x = d_x \cdot v^{x+1} = (l_x - l_{x+1}) \cdot v^{x+1} = q_x \cdot v \cdot D_x = \frac{q_x}{1+i} \cdot D_x$$

$$M_x = C_x + \ldots + C_{w'}$$

l_x may b e calculated from q_x -numbers as follows: $l_{x+1} = (1 - q_x) \cdot l_x$.

The annuity will be

$$\ddot{a}_{x:n]} = \frac{N_x - N_{x+n}}{D_x}$$

where

n is duration of annuity and
w ' is last age of tables

For a special model described later in chapter 8.2 we define as above but for interest rate i':

$$v' = \frac{1}{1+i'}$$

$$D'_x = l_x \cdot (v')^x$$

$$N'_x = \sum_{i=1}^{w} D'_i$$

$$\ddot{a}'_{x:n]} = \frac{N'_x - N'_{x+n}}{D'_x}$$

Define also monthly N_x-numbers as follows:

$$N_x^{(12)} = \frac{1}{12} \cdot \sum_{i=x}^{w} D_i^{(12)}$$

where $D_i^{(12)}$ is monthly D_x -number

5.3 Continuous model

In a continuous model l_x is defined using continuous force of mortality[16]:

$$l_x = l_0 \cdot e^{-\int_0^x \mu_s ds}$$

From this we may calculate the q_x-numbers as in discrete case.

$$q_x = 1 - \frac{l_{x+1}}{l_x} = 1 - \frac{l_0 \cdot e^{-\int_0^{x+1} \mu_s ds}}{l_0 \cdot e^{-\int_0^{x} \mu_s ds}} = 1 - e^{-\int_0^{x+1} \mu_s ds + \int_0^{x} \mu_s ds} = 1 - e^{-\int_x^{x+1} \mu_s ds}$$

Several mortality models can be used. [17] Later on I will assume that the functions have the required derivatives and integrals.

In Finland the risk functions are continuous and force of mortality is defined by the Makeham model. [18]

By using the continuous model it is possible to calculate a risk for any period. We consider first the annual case, but later also the monthly case is discussed.

By using an Euler summation we get annual representations between the continuous model and discrete model:

$$\overline{N}_x = N_x - D_x \cdot \left[\frac{1}{2} + \frac{1}{12} \cdot (\ln(1+i) + \mu_x) \right]$$

$$\overline{M}_x = D_x - \ln(1+i) \cdot \overline{N}_x$$

where D_x and N_x are calculated as in the discrete case. These continuity correction formulas are used also in the Finnish technical bases.[19] Neill writes that it is typical to use a shorter approximation:

$$\overline{N}_x = N_x - \frac{1}{2} \cdot D_x$$ [20]

[16] See about force of mortality e.g. Gerber, pp. 16 – 17, pp. 40 – 46 and Bowers et al. pp. 55 – 58.
[17] See Gerber, p. 17 – 18, Bowers et al. pp. 77 – 79 and Pesonen et al. pp. 46 – 47.
[18] See Vakuutusmatemaatikon tutkintolautakunta, Yksilöllisen henkivakuutuksen laskuperusteet 1.3.1.
[19] Ibid. Yksilöllisen henkivakuutuksen vakuutusmaksujen laskukaavat 1.11 and see Neill (3.2.1, 3.2.8, 3.3.3) p. 78, 81, 102 and Pesonen et al., pp. 66 – 67,
[20] Neill p. 78

The continuous examples that I will give later are based on the first formulas.

I concentrate on the discrete model and describe separately the behavior of the continuous model. Note that a bar above the basic symbol denotes continuous actuarial functions.

5.4 Accumulation and discount factors

In this paper I use terms "accumulation factor" and "discount factor" also for cases where not only the interest but also mortality is taken into account.

The accumulation factor including effect of interest and mortality can be written as follows:

$$\frac{D_{x+t}}{D_{x+t+1}} = \frac{1+i}{1-q_{x+t}} = 1 + \frac{i}{1-q_{x+t}} + \frac{q_{x+t}}{1-q_{x+t}}$$

The results is found as follows:

$$\frac{D_{x+t}}{D_{x+t+1}} = \frac{l_{x+t} \cdot v^{x+t}}{l_{x+t+1} \cdot v^{x+t+1}} = \frac{l_{x+t}}{l_{x+t+1} \cdot v} = \frac{1+i}{\frac{l_{x+t+1}}{l_{x+t}}} = \frac{1+i}{1-q_{x+t}} = 1 + \left(\frac{1+i}{1-q_{x+t}} - 1 \right) = 1 + \frac{1+i-1+q_{x+t}}{1-q_{x+t}}$$

$$= 1 + \frac{i+q_{x+t}}{1-q_{x+t}} = 1 + \frac{i}{1-q_{x+t}} + \frac{q_{x+t}}{1-q_{x+t}}$$

Sometimes also the respective discount factor is needed. It is equal to
$\frac{D_{x+t+1}}{D_{x+t}} = \frac{1-q_{x+t}}{1+i} = 1 - \frac{i}{1+i} - \frac{q_{x+t}}{1+i}$.

In the continuous case the discount factor may also be expressed as follows:

$$\frac{D_x}{D_{x+1}} = \frac{l_0 \cdot (1+i)^{-x} \cdot e^{-\int_0^x \mu_s ds}}{l_0 \cdot (1+i)^{-(x+1)} \cdot e^{-\int_0^{x+1} \mu_s ds}} = \frac{1}{1+i} \cdot e^{-\int_x^{x+1} \mu_s ds}$$

6 GENERAL CONVENTIONAL MODEL

6.1 Premium

Let us consider the following general premium model:

$$B_{x:k]} = \frac{A_{x:w}(*)}{\bar{a}} \cdot S_x$$

where

$A_{x:w}(*)$	= net single premium coefficient depending on the product (*) in question
S_x	= sum insured at time x for a period of n years
$\bar{a}$	= 1, for single premium
	= $\ddot{a}_{x:k]}$ for annual premiums for a period of k years

In this chapter 6 I assume that S_x and $B_{x:k]}$ are constants through insurance and payment periods. In the examples below I have used for each period the value in the beginning of the period. So, $S_{x+t+1} = S_{x+t}$ and $B_{x+t+1:k-1]} = B_{x+t:k]}$. As I later in chapter 11.1 will describe, the initial values will be used also in cases when the policy period is apportioned among periods where sum insured and premium are constants.[21] Then, for each such period these formulas will be valid.

6.2 Reserve

Reserve for sum insured S_{x+t} at time x+t in this case would be:

$$V_{x+t} = A_{x+t:w}(*) \cdot S_{x+t} - B_{x+t:k-t]} \cdot \ddot{a}_{x+t:k-t]}.$$

6.3 Discrete models

6.3.1 Pure endowment single premium

In this example the reserve for sum insured S_{x+t} at moment x+t is equal to

$$V_{x+t} = \frac{D_w}{D_{x+t}} \cdot S_{x+t}.$$ [22]

[21] See chapter 11.1

[22] See Schmidt, example 5.4.5 (5), p. 126.

After one year the reserve is equal to

$$V_{x+t+1} = \frac{D_w}{D_{x+t+1}} \cdot S_{x+t} = \frac{D_{x+t}}{D_{x+t+1}} \cdot \frac{D_w}{D_{x+t}} \cdot S_{x+t} = \frac{D_{x+t}}{D_{x+t+1}} \cdot V_{x+t}$$
$$= V_{x+t} + \frac{i + q_{x+t}}{1 - q_{x+t}} \cdot V_{x+t} = V_{x+t} + \frac{i}{1 - q_{x+t}} \cdot V_{x+t} + \frac{q_{x+t}}{1 - q_{x+t}} \cdot V_{x+t}$$

The result is natural because it shows that the reserve is the previous reserve corrected by interest increase and mortality compensation.[23]

Assuming first that mortality $q_{x+t} = 0$ and then that guaranteed interest i = 0, we obtain the following results:

1) $\frac{i}{1 - q_{x+t}} \cdot V_{x+t}$ is the effect of guaranteed interest increase

2) $\frac{q_{x+t}}{1 - q_{x+t}} \cdot V_{x+t}$ is the compensation due to mortality

6.3.2 Term life single premium

In this example the reserve for sum S_{x+t} at moment x+t is equal to

$$V_{x+t} = \frac{M_{x+t} - M_w}{D_{x+t}} \cdot S_{x+t} .$$ [24]

So, at the end of the next year the reserve formulas are as follows:

$$V_{x+t+1} = \frac{M_{x+t+1} - M_w}{D_{x+t+1}} \cdot S_{x+t} = \frac{M_{x+t} - M_w - C_{x+t}}{D_{x+t+1}} \cdot S_{x+t}$$
$$= \frac{D_{x+t}}{D_{x+t+1}} \cdot \frac{M_{x+t} - M_w}{D_{x+t}} \cdot S_{x+t} - \frac{C_{x+t}}{D_{x+t+1}} \cdot S_{x+t} = \frac{D_{x+t}}{D_{x+t+1}} \cdot V_{x+t} - \frac{d_{x+t} \cdot v^{x+t+1}}{l_{x+t+1} \cdot v^{x+t+1}} \cdot S_{x+t}$$
$$= \frac{1+i}{1 - q_{x+t}} \cdot V_{x+t} - \frac{l_{x+t} - l_{x+t+1}}{l_{x+t+1}} \cdot S_{x+t} = \frac{1+i}{1 - q_{x+t}} \cdot V_{x+t} + \left(1 - \frac{l_{x+t}}{l_{x+t+1}}\right) \cdot S_{x+t}$$
$$= \frac{1+i}{1 - q_{x+t}} \cdot V_{x+t} + \left(1 - \frac{1}{1 - q_{x+t}}\right) \cdot S_{x+t}$$

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[23] I have earlier mentioned that recursive formulas can be used to describe the methods used in the universal life product. This is an example of it. In practical solutions recursive formulas can always be found, see Koller, pp. 49 – 51.

[24] See Gerber A.4.5, p. 122 and Schmidt, example 5.4.4 (2), p. 126.

$$= \frac{1+i}{1-q_{x+t}} \cdot V_{x+t} - \frac{q_{x+t}}{1-q_{x+t}} \cdot S_{x+t} = V_{x+t} + \frac{i+q_{x+t}}{1-q_{x+t}} \cdot V_{x+t} - \frac{q_{x+t}}{1-q_{x+t}} \cdot S_{x+t}$$

$$= V_{x+t} + \frac{i}{1-q_{x+t}} \cdot V_{x+t} + \frac{q_{x+t}}{1-q_{x+t}} \cdot V_{x+t} - \frac{q_{x+t}}{1-q_{x+t}} \cdot S_{x+t}$$

$$= V_{x+t} + \frac{i}{1-q_{x+t}} \cdot V_{x+t} - \frac{q_{x+t}}{1-q_{x+t}} \cdot (S_{x+t} - V_{x+t})$$

From the second last equation we may see that the policy is entitled to compensation as in the pure endowment case and risk is charged from the whole risk sum.

However, the last equation shows that if we consider the net effect of the mortality, then there is no mortality compensation but only the positive risk sum is charged and $\frac{q_{x+t}}{1-q_{x+t}} \cdot (S_{x+t} - V_{x+t})$ is the risk charge at the end of the year.

It is also possible to calculate the charge in the beginning of the year. In this case the charge is equal to

$$-\frac{1-q_{x+t}}{1+i} \cdot \frac{q_{x+t}}{1-q_{x+t}} \cdot (S_{x+t} - V_{x+t}) = -\frac{q_{x+t}}{1+i} \cdot (S_{x+t} - V_{x+t}) .$$

In this case the reserve formula is as follows:

$$V_{x+t+1} = V_{x+t} + \frac{i}{1-q_{x+t}} \cdot V_{x+t} - \frac{q_{x+t}}{1-q_{x+t}} \cdot (S_{x+t} - V_{x+t})$$

$$= V_{x+t} + \frac{i+q_{x+t}}{1-q_{x+t}} \cdot V_{x+t} - \frac{q_{x+t}}{1-q_{x+t}} \cdot S_{x+t}$$

$$+ \frac{q_{x+t}}{1-q_{x+t}} \cdot \frac{1-q_{x+t}}{1+i} \cdot (S_{x+t} - V_{x+t}) - \frac{q_{x+t}}{1+i} \cdot (S_{x+t} - V_{x+t})$$

$$= V_{x+t} + \frac{i+q_{x+t}}{1-q_{x+t}} \cdot V_{x+t} - \frac{q_{x+t}}{1-q_{x+t}} \cdot \left(1 - \frac{1-q_{x+t}}{1+i}\right) \cdot (S_{x+t} - V_{x+t}) - \frac{q_{x+t}}{1+i} \cdot (S_{x+t} - V_{x+t})$$

$$= V_{x+t} + \frac{i+q_{x+t}}{1-q_{x+t}} \cdot V_{x+t} - \frac{i+q_{x+t}}{1-q_{x+t}} \cdot \frac{q_{x+t}}{1+i} \cdot (S_{x+t} - V_{x+t}) - \frac{q_{x+t}}{1+i} \cdot (S_{x+t} - V_{x+t})$$

$$= V_{x+t} + \frac{i+q_{x+t}}{1-q_{x+t}} \cdot \left(V_{x+t} - \frac{q_{x+t}}{1+i} \cdot (S_{x+t} - V_{x+t})\right) - \frac{q_{x+t}}{1+i} \cdot (S_{x+t} - V_{x+t})$$

So, in this formula the discounted mortality charge is decreased from the initial reserve before the interest and compensation calculations. The result is the same, but in addition both the mortality charge and the interest yield are smaller.

When deriving the formulas I mostly use the mortality at the end of a year, because the denominators of interest and mortality are the same and so the derivation of the formulas is easier. Also, as I will describe later in chapter 13.3, in some cases it is wise to start the calculation of monthly charges from the values at the end of an insurance year.

6.3.3 Annuity

In this example the reserve for sum E_{x+t} at moment x+t is equal to

$$V_{x+t} = \frac{N_{x+t} - N_w}{D_{x+t}} \cdot E_{x+t}\,.$$ [25]

Thus,

$$V_{x+t+1} = \frac{N_{x+t+1} - N_w}{D_{x+t+1}} \cdot E_{x+t} = \frac{N_{x+t} - N_w - D_{x+t}}{D_{x+t+1}} \cdot E_{x+t} = \frac{D_{x+t}}{D_{x+t+1}} \cdot \frac{N_{x+t} - N_w}{D_{x+t}} \cdot E_{x+t} - \frac{D_{x+t}}{D_{x+t+1}} \cdot E_{x+t}$$
$$= \frac{D_{x+t}}{D_{x+t+1}} \cdot (V_{x+t} - E_{x+t}) = \frac{1+i}{1-q_{x+t}} \cdot (V_{x+t} - E_{x+t})$$
$$= V_{x+t} - E_{x+t} + \frac{i}{1-q_{x+t}} \cdot (V_{x+t} - E_{x+t}) + \frac{q_{x+t}}{1-q_{x+t}} \cdot (V_{x+t} - E_{x+t})$$

Because the annuity is paid in advance, interest and mortality compensation are added to the difference of the reserve and the annuity.

6.3.4 Effect of premiums

In this example we analyze the effect of payment $B_{x+t:k-t]}$ for the reserve. If $A_{x+t:w}(*)$ is any single net premium coefficient at moment x+t and $B_{x+t:k-t]}$ annual payment paid for k-t years, then reserve at moment x+t is equal to

$$V_{x+t} = A_{x+t:w}(*) - B_{x+t:k-t]} \cdot \ddot{a}_{x+t:k-t]}\,.$$

Thus,

$$V_{x+t+1} = A_{x+t+1:w}(*) - B_{x+t:k-t]} \cdot \ddot{a}_{x+t+1:k-t-1]}$$
$$= A_{x+t+1:w}(*) - B_{x+t:k-t]} \cdot \frac{N_{x+t+1} - N_{x+k}}{D_{x+t+1}}$$
$$= A_{x+t+1:w}(*) - B_{x+t:k-t]} \cdot \frac{N_{x+t} - N_{x+k} - D_{x+t}}{D_{x+t+1}}$$
$$= A_{x+t+1:w}(*) - B_{x+t:k-t]} \cdot \left(\frac{D_{x+t}}{D_{x+t+1}} \cdot \frac{N_{x+t} - N_{x+k}}{D_{x+t}} - \frac{D_{x+t}}{D_{x+t+1}} \right)$$
$$= A_{x+t+1:w}(*) - \frac{D_{x+t}}{D_{x+t+1}} \cdot B_{x+t:k-t]} \cdot (\ddot{a}_{x+t:k-t]} - 1)$$
$$= A_{x+t+1:w}(*) - \frac{1+i}{1-q_{x+t}} \cdot B_{x+t:k-t]} \cdot (\ddot{a}_{x+t:k-t]} - 1)$$
$$= A_{x+t+1:w}(*) - B_{x+t:k-t]} \cdot (\ddot{a}_{x+t:k-t]} - 1) \cdot \left(1 + \frac{i}{1-q_{x+t}} + \frac{q_{x+t}}{1-q_{x+t}}\right)$$

[25] See Gerber A.3.6, p. 121 and Schmidt, example 5.4.5 (2), p. 126.

Here $B_{x+t:k-t]} + \frac{i}{1-q_{x+t}} \cdot B_{x+t:k-t]} + \frac{q_{x+t}}{1-q_{x+t}} \cdot B_{x+t:k-t]}$ is the effect of the payment. Because the premium is paid in advance, also interest and mortality compensation are added to the reserve.

As we may see, the effect of the payments is similar to the effect of annuities. Payments are just positive whereas annuities negative cash flow.

6.3.5 Deferred annuity

In this example the annuity has been deferred by m years. Then the reserve for sum E_{x+t} at moment x+t is equal to

$$V_{x+t} = \frac{N_{x+t+m} - N_w}{D_{x+t}} \cdot E_{x+t} .$$

If m > 0, then

$$V_{x+t+1} = \frac{N_{x+t+m} - N_w}{D_{x+t+1}} \cdot E_{x+t} = \frac{D_{x+t}}{D_{x+t+1}} \cdot \frac{N_{x+t+m} - N_w}{D_{x+t}} \cdot E_{x+t} = \frac{D_{x+t}}{D_{x+t+1}} \cdot V_{x+t} .$$

From the results of pure endowment case we realize that during deferred period the reserve is equal to

$$V_{x+t+1} = V_{x+t} + \frac{i}{1-q_{x+t}} \cdot V_{x+t} + \frac{q_{x+t}}{1-q_{x+t}} \cdot V_{x+t} .$$

The result is clear. The deferred period acts like a paid-up policy.

6.3.6 Summary of general discrete model

The reserve may be calculated by the following formula:[26]

$$V_{x+t+1} = V_{x+t}$$
$$+ B_{x+t:k-t]} - E_{x+t} + \frac{i}{1-q_{x+t}} \cdot (V_{x+t} + B_{x+t:k-t]} - E_{x+t}) - \frac{q_{x+t}}{1-q_{x+t}} \cdot (S_{x+t} - (V_{x+t} + B_{x+t:k-t]} - E_{x+t}))$$

The last term can be positive or negative depending on whether the risk sum is positive or negative.

[26] This general result without division to the components and slightly differently expressed has been proofed also in the literature. The proofs do not directly show the relationship with the commutation numbers. See e.g. Neill (4.4.1), p. 124 and Bowers et al. (8.3.10) p. 235, Schmidt ((5.5.9) p. 124 and Gerber (6.3.4) p. 61.

The different components are as follows:

- annual premium $B_{x+t:k-t]}$
- annual annuity E_{x+t}
- interest $\frac{i}{1-q_{x+t}} \cdot (V_{x+t} + B_{x+t:k-t]} - E_{x+t})$
- mortality $-\frac{q_{x+t}}{1-q_{x+t}} \cdot \left[S_{x+t} - (V_{x+t} + B_{x+t:k-t]} - E_{x+t})\right]^{+}$
- compensation $-\frac{q_{x+t}}{1-q_{x+t}} \cdot \left[S_{x+t} - (V_{x+t} + B_{x+t:k-t]} - E_{x+t})\right]^{-}$

It is also possible to decrease the mortality charge in the beginning of the year. The formulas for this were given in chapter 6.3.2.

6.4 Continuous models[27]

6.4.1 General

If the continuous model does not have continuity correction, then the formulas defined in the discrete model apply. Below I describe the effects of the continuity correction as defined in chapter 5.3.

6.4.2 Continuous annuity

Using continuity correction, the reserve of the annuity for sum E_{x+t} at moment x+t is equal to:

$$V_{x+t} = \frac{\overline{N}_{x+t} - \overline{N}_w}{D_{x+t}} \cdot E_{x+t}$$

$$= \frac{N_{x+t} - D_{x+t} \cdot \left[\frac{1}{2} + \frac{1}{12} \cdot \left(\ln(1+i) + \mu_{x+t}\right)\right] - N_w + D_w \cdot \left[\frac{1}{2} + \frac{1}{12} \cdot \left(\ln(1+i) + \mu_w\right)\right]}{D_{x+t}} \cdot E_{x+t}$$

$$= \left\{\frac{N_{x+t} - N_w}{D_{x+t}} - \left[\frac{1}{2} + \frac{1}{12} \cdot \left(\ln(1+i) + \mu_{x+t}\right)\right] + \frac{D_w}{D_{x+t}} \cdot \left[\frac{1}{2} + \frac{1}{12} \cdot \left(\ln(1+i) + \mu_w\right)\right]\right\} \cdot E_{x+t}$$

Thus using the results from discrete model we get a value at the end of the year:

$$V_{x+t+1}$$

$$= \left\{\frac{N_{x+t+1} - N_w}{D_{x+t+1}} - \left[\frac{1}{2} + \frac{1}{12} \cdot \left(\ln(1+i) + \mu_{x+t+1}\right)\right] + \frac{D_w}{D_{x+t+1}} \cdot \left[\frac{1}{2} + \frac{1}{12} \cdot \left(\ln(1+i) + \mu_w\right)\right]\right\} \cdot E_{x+t}$$

[27] Some formulas have been revised after first edition of the document.

$$= \frac{D_{x+t}}{D_{x+t+1}} \cdot \left\{ \frac{N_{x+t} - N_w}{D_{x+t}} - 1 - \left[\frac{1}{2} + \frac{1}{12} \cdot \left(\ln(1+i) + \mu_{x+t} \right) \right] + \frac{D_w}{D_{x+t}} \cdot \left[\frac{1}{2} + \frac{1}{12} \cdot \left(\ln(1+i) + \mu_w \right) \right] \right\} \cdot E_{x+t}$$

$$- \frac{D_{x+t}}{D_{x+t+1}} \cdot \left\{ - \left[\frac{1}{2} + \frac{1}{12} \cdot \left(\ln(1+i) + \mu_{x+t} \right) \right] + \frac{D_w}{D_{x+t}} \cdot \left[\frac{1}{2} + \frac{1}{12} \cdot \left(\ln(1+i) + \mu_w \right) \right] \right\} \cdot E_{x+t}$$

$$- \left\{ \left[\frac{1}{2} + \frac{1}{12} \cdot \left(\ln(1+i) + \mu_{x+t+1} \right) \right] + \frac{D_w}{D_{x+t+1}} \cdot \left[\frac{1}{2} + \frac{1}{12} \cdot \left(\ln(1+i) + \mu_w \right) \right] \right\} \cdot E_{x+t}$$

$$= \frac{D_{x+t}}{D_{x+t+1}} \cdot (V_{x+t}^A - E_{x+t})$$

$$+ \frac{D_{x+t}}{D_{x+t+1}} \cdot \left\{ \left[\frac{1}{2} + \frac{1}{12} \cdot \left(\ln(1+i) + \mu_{x+t} \right) \right] - \frac{D_{x+t+1}}{D_{x+t}} \cdot \left[\frac{1}{2} + \frac{1}{12} \cdot \left(\ln(1+i) + \mu_{x+t+1} \right) \right] \right\} \cdot E_{x+t}$$

$$= \frac{1+i}{1-q_{x+t}} \cdot (V_{x+t} - E_{x+t})$$

$$+ \frac{1+i}{1-q_{x+t}} \cdot \left\{ \frac{1}{2} + \frac{1}{12} \cdot \left(\ln(1+i) + \mu_{x+t} \right) - \frac{1-q_{x+t}}{1+i} \cdot \left[\frac{1}{2} + \frac{1}{12} \cdot \left(\ln(1+i) + \mu_{x+t+1} \right) \right] \right\} \cdot E_{x+t}$$

Compared to the discrete model there is continuity correction in the beginning of the year equal to

$$\left\{ \frac{1}{2} + \frac{1}{12} \cdot \left(\ln(1+i) + \mu_{x+t} \right) - \frac{1-q_{x+t}}{1+i} \cdot \left[\frac{1}{2} + \frac{1}{12} \cdot \left(\ln(1+i) + \mu_{x+t+1} \right) \right] \right\} \cdot E_{x+t}$$

$$= \left\{ \frac{1}{2} + \frac{1}{12} \cdot \left(\ln(1+i) + \mu_{x+t} \right) - \left(1 - \frac{i+q_{x+t}}{1+i} \right) \cdot \left[\frac{1}{2} + \frac{1}{12} \cdot \left(\ln(1+i) + \mu_{x+t+1} \right) \right] \right\} \cdot E_{x+t}$$

$$= \left\{ \frac{1}{2} + \frac{1}{12} \cdot \left(\ln(1+i) + \mu_{x+t} \right) - \frac{1}{2} - \frac{1}{12} \cdot (\ln(1+i) + \mu_{x+t+1}) + \frac{i+q_{x+t}}{1+i} \cdot \left[\frac{1}{2} + \frac{1}{12} \cdot \left(\ln(1+i) + \mu_{x+t+1} \right) \right] \right\} \cdot E_{x+t}$$

$$= \left\{ \frac{1}{12} \cdot (\mu_{x+t} - \mu_{x+t+1}) + \frac{i+q_{x+t}}{1+i} \cdot \left[\frac{1}{2} + \frac{1}{12} \cdot \left(\ln(1+i) + \mu_{x+t+1} \right) \right] \right\} \cdot E_{x+t}$$

Reserves are corrected each time the annuity is paid out. The correction increases the reserve as described below in the summary.

6.4.3 Continuous payments

Similar to the calculation for continuous annuities, the payment correction in the beginning of the year is equal to

$$- \left\{ \frac{1}{12} \cdot (\mu_{x+t} - \mu_{x+t+1}) + \frac{i+q_{x+t}}{1+i} \cdot \left[\frac{1}{2} + \frac{1}{12} \cdot \left(\ln(1+i) + \mu_{x+t+1} \right) \right] \right\} \cdot \cdot B_{x+t}$$

Reserves are corrected each time a payment has been collected. The correction decreases the reserve as described below in the summary.

6.4.4 Continuous term life

In this example the reserve for sum S_{x+t} at moment x+t is equal to

$$V_{x+t} = \frac{\overline{M}_{x+t} - \overline{M}_w}{D_{x+t}} \cdot S_{x+t}$$
$$= \frac{D_{x+t} - \ln(1+i)\cdot \overline{N}_{x+t} - D_w - \ln(1+i)\cdot \overline{N}_w}{D_{x+t}} \cdot S_{x+t}$$
$$= \left(\frac{D_{x+t} - D_w}{D_{x+t}} - \ln(1+i)\cdot \frac{\overline{N}_{x+t} - \overline{N}_w}{D_{x+t}} \right) \cdot S_{x+t}$$

Using the results from annuity and the discrete model we obtain:

$$V_{x+t+1}$$
$$= \left(\frac{D_{x+t+1} - D_w}{D_{x+t+1}} - \ln(1+i)\cdot \frac{\overline{N}_{x+t+1} - \overline{N}_w}{D_{x+t+1}} \right) \cdot S_{x+t}$$
$$= \left(\frac{D_{x+t} - D_w}{D_{x+t+1}} - \ln(1+i)\cdot \frac{\overline{N}_{x+t+1} - \overline{N}_w}{D_{x+t+1}} \right) \cdot S_{x+t} + \left(1 - \frac{D_{x+t}}{D_{x+t+1}} - \ln(1+i)\cdot \frac{\overline{N}_{x+t+1} - \overline{N}_{x+t}}{D_{x+t+1}} \right) \cdot S_{x+t}$$
$$= \frac{D_{x+t}}{D_{x+t+1}} \cdot V_{x+t} + \left(1 - \frac{D_{x+t}}{D_{x+t+1}} - \ln(1+i)\cdot \frac{\overline{N}_{x+t+1} - \overline{N}_{x+t}}{D_{x+t+1}} \right) \cdot S_{x+t}$$
$$= V_{x+t} + \frac{i + q_{x+t}}{1 - q_{x+t}} \cdot V_{x+t} - \frac{q_{x+t}}{1 - q_{x+t}} \cdot S_{x+t} + \left(1 - 1 - \frac{i + q_{x+t}}{1 - q_{x+t}} - \ln(1+i)\cdot \frac{\overline{N}_{x+t+1} - \overline{N}_{x+t}}{D_{x+t+1}} + \frac{q_{x+t}}{1 - q_{x+t}} \right) \cdot S_{x+t}$$
$$= V_{x+t} + \frac{i + q_{x+t}}{1 - q_{x+t}} \cdot V_{x+t} - \frac{q_{x+t}}{1 - q_{x+t}} \cdot S_{x+t} - \left(\frac{i}{1 - q_{x+t}} + \ln(1+i)\cdot \frac{\overline{N}_{x+t+1} - \overline{N}_{x+t}}{D_{x+t+1}} \right) \cdot S_{x+t}$$

So, the continuity correction of the mortality in the beginning of the year is equal to

$$-\frac{1 - q_{x+t}}{1+i} \cdot \left(\frac{i}{1 - q_{x+t}} + \ln(1+i)\cdot \frac{\overline{N}_{x+t+1} - \overline{N}_{x+t}}{D_{x+t+1}} \right) \cdot S_{x+t}$$
$$= -\left(\frac{i}{1 - q_{x+t}} + \ln(1+i)\cdot \frac{\overline{N}_{x+t+1} - \overline{N}_{x+t}}{D_{x+t+1}} \right) \cdot S_{x+t}$$
$$= -\left(\frac{i}{1 - q_{x+t}} + \ln(1+i)\cdot \frac{N_{x+t+1} - D_{x+t+1}\cdot \left[\frac{1}{2} + \frac{1}{12}\cdot \left(\ln(1+i) + \mu_{x+t+1} \right) \right] - N_{x+t} + D_{x+t}\cdot \left[\frac{1}{2} + \frac{1}{12}\cdot \left(\ln(1+i) + \mu_{x+t} \right) \right]}{D_{x+t+1}} \right) \cdot S_{x+t}$$
$$= -\left(\frac{i}{1 - q_{x+t}} + \ln(1+i)\cdot \left\{ \frac{N_{x+t+1} - N_{x+t}}{D_{x+t+1}} - \frac{D_{x+t+1}}{D_{x+t+1}} \cdot \left[\frac{1}{2} + \frac{1}{12}\cdot \left(\ln(1+i) + \mu_{x+t+1} \right) \right] + \frac{D_{x+t}}{D_{x+t+1}} \cdot \left[\frac{1}{2} + \frac{1}{12}\cdot \left(\ln(1+i) + \mu_{x+t} \right) \right] \right\} \right) \cdot S_{x+t}$$

$$= -\left(\frac{i}{1-q_{x+t}} + \ln(1+i) \cdot \left\{ -\frac{1+i}{1-q_{x+t}} - \left[\frac{1}{2} + \frac{1}{12} \cdot \left(\ln(1+i) + \mu_{x+t+1} \right) \right] + \frac{1+i}{1-q_{x+t}} \cdot \left[\frac{1}{2} + \frac{1}{12} \cdot \left(\ln(1+i) + \mu_{x+t} \right) \right] \right\} \right) \cdot S_{x+t}$$

$$= -\left(\frac{i}{1-q_{x+t}} + \ln(1+i) \cdot \left\{ -\frac{1+i}{1-q_{x+t}} - \frac{1+i}{1-q_{x+t}} \cdot \left(\frac{1-q_{x+t}}{1+i} \cdot \left[\frac{1}{2} + \frac{1}{12} \cdot \left(\ln(1+i) + \mu_{x+t+1} \right) \right] + \frac{1}{2} + \frac{1}{12} \cdot \left(\ln(1+i) + \mu_{x+t} \right) \right) \right\} \right) \cdot S_{x+t}$$

Here we may use the results of the proof of the continuous annuity case and find out that at the end of the year this is equal to

$$\left(-\frac{i}{1-q_{x+t}} + \ln(1+i) \cdot \left\{ \frac{1+i}{1-q_{x+t}} \cdot \left[1 - \frac{1}{12} \cdot (\mu_{x+t} - \mu_{x+t+1}) \right] - \frac{1}{2} + \frac{1}{12} \cdot \left(\ln(1+i) + \mu_{x+t+1} \right) \right\} \right) \cdot S_{x+t}$$

This correction corrects the reserves each time the mortality is charged.

If the discrete mortality value is calculated in the beginning of the year, the discounted formula is equal to

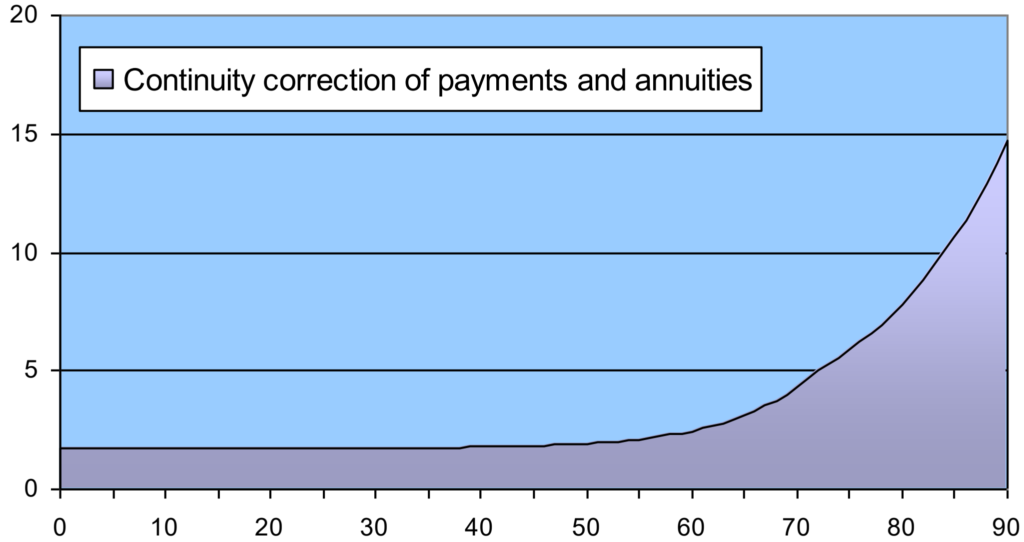
$$\left(-\frac{i}{1+i} + \ln(1+i) \cdot \left\{ 1 - \frac{1}{12} \cdot (\mu_{x+t} - \mu_{x+t+1}) - \frac{i+q_{x+t}}{1+i} \cdot \left[\frac{1}{2} + \frac{1}{12} \cdot \left(\ln(1+i) + \mu_{x+t+1} \right) \right] \right\} \right) \cdot S_{x+t}$$

6.4.5 Summary of general continuous model

In the continuous model it is possible to use continuity corrections for the discrete values. They act like loadings, which will be discussed more thoroughly in the next chapter. For example payment corrections are taken into account only when payments are paid.

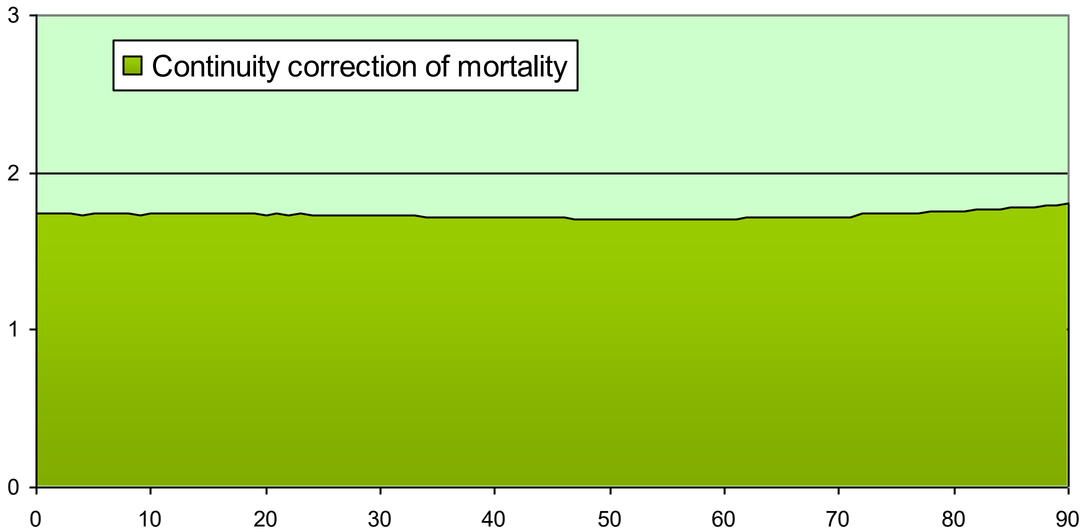
The payment and continuity correction related to it should be separate, because if the actual payment is not shown to the policyholder, then the policyholder cannot be sure that the payment has been taken into account. This is the same also in annuity case.

The correction related to annuities and payments for interest rate 3,5 % is shown in picture 6.1. The correction is almost constant until age 40 and then rises to 13,8 % until age 90.



Picture 6.1 Continuity correction effect of payments and annuities (%) for interest rate 3,5 % at the end of the year

Mortality charges are charged each year. The continuity correction of the mortality can be added to the value of the discrete model because it has originally defined to be a component of the mortality. The correction does not vary great deal, especially if charged at the end of the year as shown in picture 6.2. During the same period as above the correction ranges from 1,71 % to 1,80 % with lowest value at age 56.



Picture 6.2 Continuity correction effect of mortality (%) for interest rate 3,5 % if charged at the end of the year

If the charges were charged in the beginning of the year, then the discount factor should have been taken into account. The factor decreases the value the more the older a person is.

One option is to add also the payment and annuity correction to this value. However, it can be problematic if the mortality charge becomes positive due to the correction.

As a summary, the reserve may be calculated by

$$
\begin{aligned}
V_{x+t+1} &= V_{x+t} \\
&+ B_{x+t:k-t]} - E_{x+t} + \frac{i}{1-q_{x+t}} \cdot (V^A_{x+t} + B_{x+t:k-t]} - E_{x+t}) \\
&- \frac{q_{x+t}}{1-q_{x+t}} \cdot (S_{x+t} - (V^A_{x+t} + B_{x+t:k-t]} - E_{x+t})) + B^C_{x+t} + E^C_{x+t} + Q^C_{x+t}
\end{aligned}
$$

where
B^C_{x+t} is the correction related to payments
E^C_{x+t} is the correction related to annuities
Q^C_{x+t} is the correction related to mortality charges
the other notations are the same as in discrete model

The second last term can be positive or negative depending on whether there exists a positive or a negative risk sum.

The different correction components are as follows:
- payment correction

$$-\left\{\frac{1}{12}\cdot(\mu_{x+t}-\mu_{x+t+1})+\frac{i+q_{x+t}}{1+i}\cdot\left[\frac{1}{2}+\frac{1}{12}\cdot\left(\ln(1+i)+\mu_{x+t+1}\right)\right]\right\}\cdot B_{x+t}$$

- annuity correction

$$\left\{\frac{1}{12}\cdot(\mu_{x+t}-\mu_{x+t+1})+\frac{i+q_{x+t}}{1+i}\cdot\left[\frac{1}{2}+\frac{1}{12}\cdot\left(\ln(1+i)+\mu_{x+t+1}\right)\right]\right\}\cdot E_{x+t}$$

- mortality correction

$$\left(-\frac{i}{1-q_{x+t}}+\ln(1+i)\cdot\frac{1+i}{1-q_{x+t}}\cdot\left\{1-\frac{1}{12}\cdot(\mu_{x+t}-\mu_{x+t+1})-\frac{i+q_{x+t}}{1+i}\cdot\left[\frac{1}{2}+\frac{1}{12}\cdot\left(\ln(1+i)+\mu_{x+t+1}\right)\right]\right\}\right)\cdot S_{x+t}$$

7 GENERAL EXPENSE-LOADED PREMIUM MODEL

7.1 Premium

Let us consider the following general expense-loaded premium model. The model is the one proposed by Gerber.[28]

$$B_{x:k]} = \frac{A_{x:w}(*) + \alpha_1 + \gamma \cdot \ddot{a}_{x:w}}{(1-\beta)\cdot \overline{a}} \cdot S_x$$

where

$A_{x:w}(*)$	= net single premium coefficient depending on the product in concern
S_x	= sum insured at time x for a period of n years
α_1	= loading parameter in relation to the sum insured, charged in the beginning of the policy period
γ	= loading parameter in relation to the sum insured, charged annually
β	= loading parameter in relation to each gross payment, charged during the payment period
$\overline{a}$	= 1, for single premium = $\ddot{a}_{x:k]}$ for annual premiums

In this chapter 7 we still assume as we did in chapter 6.1 that S_x and $B_{x:k]}$ are constants through insurance and payment periods. So, $S_{x+t} = S_x$ and $B_{x+t:k-t]} = B_{x:k]}$. Later in chapter 11.1 it is found that this restriction is not relevant.

7.2 Reserve

Gross reserve to sum insured S_{x+t} is

$$V_{x+t} = A_{x+t:w}(*) \cdot S_{x+t} - (1-\beta)\cdot B_{x+t:k-t]} \cdot \ddot{a}_{x+t:k-t]} + \gamma \cdot \ddot{a}_{x+t:w} \cdot S_{x+t} \, .$$

[28] See Gerber, p. 104.

In this model α_1-loading is charged as a single charge in the beginning of the policy period and γ-loading is charged annually.

From the reserve formula for time x+t+1 we obtain

$$
\begin{aligned}
V_{x+t+1} &= A_{x+t+1:w}(*)\cdot S_{x+t} - (1-\beta)\cdot B_{x+t:k-t]}\cdot \ddot{a}_{x+t+1:k-t-1]} + \gamma\cdot \ddot{a}_{x+t+1:w}\cdot S_{x+t} \\
&= A_{x+t:w}(*)\cdot S_{x+t} - (1-\beta)\cdot B_{x+t:k-t]}\cdot \ddot{a}_{x+t:k-t]} + \gamma\cdot \ddot{a}_{x+t:w}\cdot S_{x+t} + (A_{x+t+1:w}(*) - A_{x+t:w}(*))\cdot S_{x+t} \\
&\quad - B_{x+t:k-t]}\cdot(\ddot{a}_{x+t+1:k-t-1]} - \ddot{a}_{x+t:k-t]}) + \gamma\cdot(\ddot{a}_{x+t+1:w} - \ddot{a}_{x+t:w})\cdot S_{x+t} + \beta\cdot B_{x+t:k-t]}\cdot(\ddot{a}_{x+t+1:k-t-1]} - \ddot{a}_{x+t:k-t]}) \\
&= V_{x+t} + (A_{x+t+1:w}(*) - A_{x+t:w}(*))\cdot S_{x+t} - B_{x+t:k-t]}\cdot(\ddot{a}_{x+t+1:k-t-1]} - \ddot{a}_{x+t:k-t]}) + \gamma\cdot(\ddot{a}_{x+t+1:w} - \ddot{a}_{x+t:w})\cdot S_{x+t} \\
&\quad + \beta\cdot B_{x+t:k-t]}\cdot(\ddot{a}_{x+t+1:k-t-1]} - \ddot{a}_{x+t:k-t]})
\end{aligned}
$$

The different components will be analyzed below.

7.3 Costs charged in the beginning of the policy period

From the premium formula we obtain the following result:

$$(1-\beta)\cdot B_{x:k]}\cdot \ddot{a}_{x:k]} = A_{x:w}(*)\cdot S_{x+t} + (\alpha_1 + \gamma\cdot \ddot{a}_{x:w})\cdot S_x$$

Calculating the reserve as t = 0 we obtain

$$V_x = A_{x:w}(*)\cdot S_x - (1-\beta)\cdot B_{x:k]}\cdot \ddot{a}_{x:k]} + \gamma\cdot \ddot{a}_{x:w}\cdot S_x$$

Replacing the result from the premium formula to this we obtain

$$
\begin{aligned}
V_x &= A_{x:w}(*)\cdot S_x - A_{x:w}(*)\cdot S_{x+t} - (\alpha_1 + \gamma\cdot \ddot{a}_{x:w})\cdot S_x + \gamma\cdot \ddot{a}_{x:w}\cdot S_x \\
&= -\alpha_1\cdot S_x
\end{aligned}
$$

So, this means that in the beginning of the policy period, an α_1-loading equal to $-\alpha_1\cdot S_x$ is charged.

7.4 Costs charged from each payment

The reserve formula for V_{x+t+1} shows that the effect of each payment is equal to $-\beta \cdot B_{x+t:\overline{k-t}|}$ and is charged at the same time as the payment. So, at the end of the year the effect is equal to $-\left(1+\frac{i}{1-q_{x+t}}+\frac{q_{x+t}}{1-q_{x+t}}\right)\cdot \beta \cdot B_{x+t:\overline{k-t}|}$.

7.5 Costs charged during the insurance policy period

The reserve formula for V_{x+t+1} shows also that the effect of γ-loading is equal to $\gamma \cdot (\ddot{a}_{x+t+1:w} - \ddot{a}_{x+t:w}) \cdot S_{x+t}$ and is charged in the beginning of an insurance year. So, at the end of the year the effect is equal to $-\left(1+\frac{i}{1-q_{x+t}}+\frac{q_{x+t}}{1-q_{x+t}}\right)\cdot \gamma \cdot S_{x+t}$.

7.6 Summary of expense-loaded premium model

The reserve may be calculated by

$$V_{x+t+1} = V_{x+t}$$
$$+ (1-\beta)\cdot B_{x+t:\overline{k-t}|} - E_{x+t} - \alpha_1 \cdot S_{x+t} + \frac{i}{1-q_{x+t}} \cdot \left[V_{x+t} + (1-\beta)\cdot B_{x+t:\overline{k-t}|} - E_{x+t} - \gamma \cdot S_{x+t}\right]$$
$$- \frac{q_{x+t}}{1-q_{x+t}} \cdot \left[S_{x+t} - (V_{x+t} + (1-\beta)\cdot B_{x+t:\overline{k-t}|} - E_{x+t} - \gamma \cdot S_{x+t})\right]$$

The last term can be positive or negative depending on whether there exists a positive or a negative risk sum.

The different components are:

- annual premium	$B_{x+t:\overline{k-t}|}$
- annual annuity	E_{x+t}
- initial costs	$-\alpha_1 \cdot S_x$
- annual administration costs	$-\gamma \cdot S_{x+t}$
- premium related costs	$-\beta \cdot B_{x+t:\overline{k-t}|}$

- interest

$$\frac{i}{1-q_{x+t}} \cdot (V_{x+t} + (1-\beta) \cdot B_{x+t:k-t]} - E_{x+t} - \gamma \cdot S_{x+t})$$

- mortality

$$-\frac{q_{x+t}}{1-q_{x+t}} \cdot \left[S_{x+t} - (V_{x+t} + (1-\beta) \cdot B_{x+t:k-t]} - E_{x+t} - \gamma \cdot S_{x+t}) \right]^{+}$$

- compensation

$$-\frac{q_{x+t}}{1-q_{x+t}} \cdot \left[S_{x+t} - (V_{x+t} + (1-\beta) \cdot B_{x+t:k-t]} - E_{x+t} - \gamma \cdot S_{x+t}) \right]^{-}$$

The values in the beginning of the year are derived in a corresponding manner.

8 EXTENSIONS TO GENERAL EXPENSE-LOADED PREMIUM MODEL

8.1 Modified premium related costs

Let us consider the following expense-loaded premium model:

$$B_{x:k]} = \frac{A_{x:w}(*) + \alpha_1 + \gamma \cdot \ddot{a}_{x:w}}{(1-\beta)\cdot \bar{a} - \alpha_2} \cdot S_x$$

The model is the same as the previously-mentioned general expense-loaded premium model, with the addition of the α_2-loading. The formula can now be written as follows:

$$B_{x:k]} = \frac{A_{x:w}(*) + \alpha_1 + \gamma \cdot \ddot{a}_{x:w}}{(1-\beta - \frac{\alpha_2}{\bar{a}})\cdot \bar{a}} \cdot S_x$$

So, the annual expense at time x+t is equal to $-\frac{\alpha_2}{\bar{a}} \cdot B_{x+t:k-t]}$ which means that $\alpha_2 \cdot B_{x+t:k-t]}$ is apportioned for the whole payment period.[29] The loading acts in the same way as β-loading.

8.2 Modified costs charged during the policy period

I have seen also the following γ-loading model been used:

$$B_{x:k]} = \frac{A_{x:w}(*) + \gamma \cdot \ddot{a}'_{x:w}}{\bar{a}} \cdot S_x$$

where $\ddot{a}'_{x:w}$ has been calculated with other interest rate i' than the other parts of the reserve.

The reserve was defined to be:

$$V_{x+t} = A_{x+t:w}(*) \cdot S_{x+t} - B_{x+t:k-t]} \cdot \ddot{a}_{x+t:k-t]} + \gamma \cdot \ddot{a}'_{x+t:w} \cdot S_{x+t}$$

I have analyzed the above equation in the general expense-loaded premium model the case where only one interest rate was used. Taking into account those results we obtain the following reserve at time x+t+1:

29 Personally I would not propose to use this kind of loading unless there is a special penalty loading for the case where the payments are terminated before the end of the policy period.

$$V_{x+t+1} = -\frac{1+i}{1-q_{x+t}} \cdot \ddot{a}'_{x+t:w} \cdot \gamma \cdot S_{x+t} + \frac{1+i'}{1-q_{x+t}} \cdot (\ddot{a}'_{x+t:w} - 1) \cdot \gamma \cdot S_{x+t}$$

Here for the part $\ddot{a}'_{x+t:w} \cdot \gamma \cdot S_{x+t}$ interest rate i and for the rest - that is $(\ddot{a}'_{x+t:w} - 1) \cdot \gamma \cdot S_{x+t}$ - interest i' is applied.

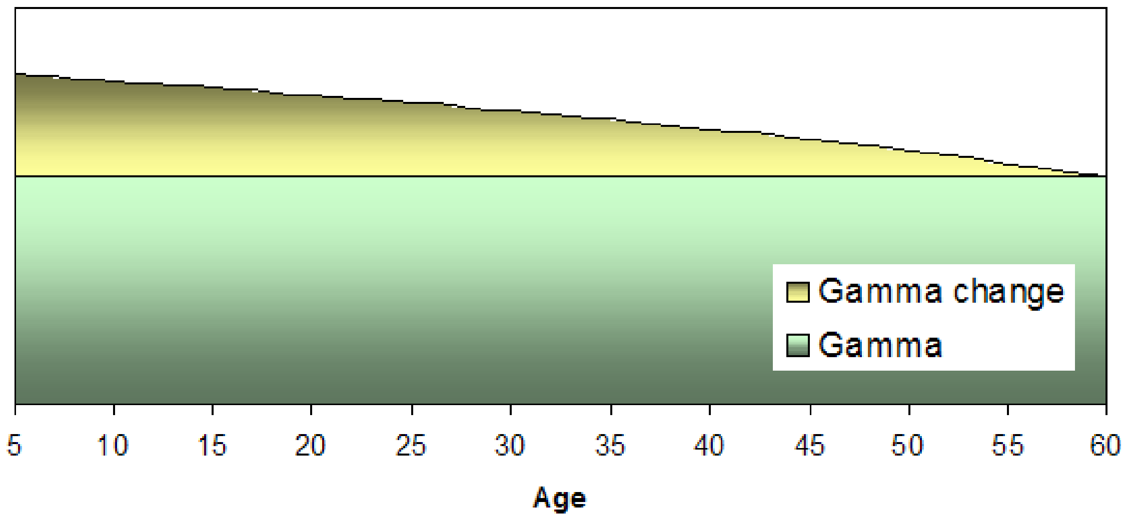
This formula may also be written as follows:

$$V_{x+t+1} = -\frac{1+i}{1-q_{x+t}} \cdot \ddot{a}'_{x+t:w} \cdot \gamma \cdot S_{x+t} + \frac{1+i'}{1-q_{x+t}} \cdot (\ddot{a}'_{x+t:w} - 1) \cdot \gamma \cdot S_{x+t}$$
$$= -\frac{1+i}{1-q_{x+t}} \cdot \gamma \cdot S_{x+t} - \frac{1+i}{1-q_{x+t}} \cdot (\ddot{a}'_{x+t:w} - 1) \cdot \gamma \cdot S_{x+t} + \frac{1+i'}{1-q_{x+t}} \cdot (\ddot{a}'_{x+t:w} - 1) \cdot \gamma \cdot S_{x+t}$$
$$= -\frac{1+i}{1-q_{x+t}} \cdot \gamma \cdot S_{x+t} - \frac{i-i'}{1-q_{x+t}} \cdot (\ddot{a}'_{x+t:w} - 1) \cdot \gamma \cdot S_{x+t}$$
$$= \frac{1+i}{1-q_{x+t}} \cdot \left[-\gamma \cdot S_{x+t} - \frac{i-i'}{1+i} \cdot (\ddot{a}'_{x+t:w} - 1) \cdot \gamma \cdot S_{x+t} \right]$$

The first term is now the same as the effect of interest and mortality for the γ-loading in the general expense-loaded premium model, and the second term displays the difference to this loading. Let us now analyze the second term discounted to the beginning of the year[30]:

$$\frac{i-i'}{1+i} \cdot (\ddot{a}'_{x+t:w} - 1) \cdot \gamma \cdot S_{x+t}$$

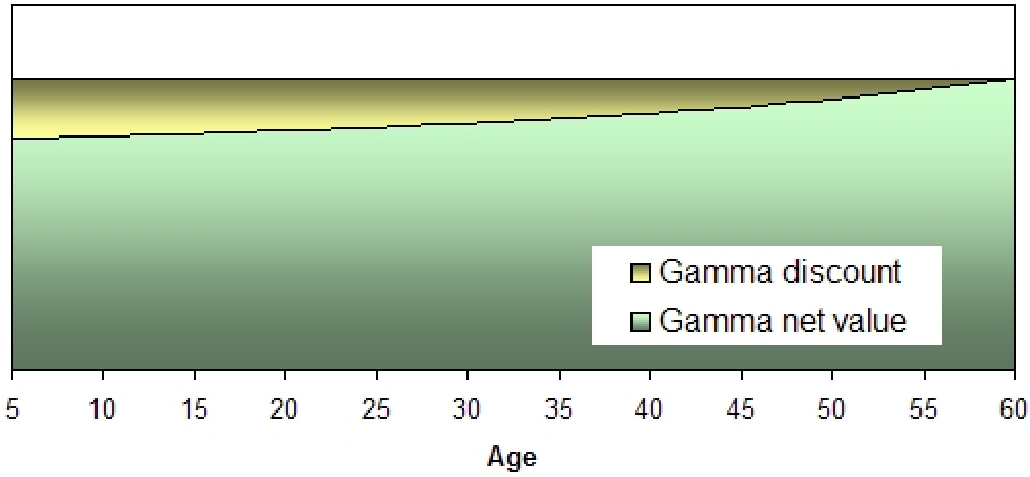
In the case where i' < i, the second term is negative. As an example, picture 8.1 shows Estonian mortality table 1997 with i = 3 % and i' = 2 % being used.



Picture 8.1. Effect of gamma loading in the case where the technical interest rate used for gamma is lower than those applied for the rest of the reserve

[30] If this is not discounted, then the curve is not so regular.

In the case where i > i', the second term is positive. As an example, picture 8.2 shows the Estonian mortality table 1997 with i = 3 % and i' = 4 % being used.



Picture 8.2. Effect of gamma loading in case the used technical interest rate for gamma is bigger than those applied for the rest of the reserve

I have chosen this as an example because it shows clearly that sometimes it is not easy to define and name the components of the premium. However, because of transparency requirements, the charge components should be shown to the customer. Below I consider some alternatives:

Let us first assume that i' < i.[31]

What type of loading is this?
- Should it be γ-loading because it is related to it? In this case the γ-loading is maybe not the same as in the price list because normally only the loading percentage is given in the price list and the effect of the interest is not clearly shown. Then there would be one component called simply "Gamma loading".
- Should it reduce the interest yield because it is caused by the lower interest rate? In this case the interest yield would sometimes be negative.
- Should it be some other loading? In this case there might be a problem if it has not been mentioned in the price list. However, the interest rate and formulas used have been mentioned in the technical bases. So, maybe it should simply be called "Gamma change".

[31] I have seen this model used, but personally I do not see any reason to use such a model.

Let us then assume that i' > i.[32]

What type of loading is this?
- Should it be γ-loading because it is related to it? This would be one option to call the net value of γ-loading simply "Gamma loading".
- Should it be discount of γ-loading because it is related to it? This would be attractive to the customer. There would be two components: "Gamma" and "Gamma discount".
- Should it increase the interest yield because it is caused by the higher interest rate? This could be one option, but it is difficult to describe the customer how the interest yield has been calculated.

I do not provide final answers, but the management of the undertaking should agree upon the name of the loading.

8.3 Loading for premiums paid several times a year

Let us denote by $B^{(m)}_{x:k]}$ a premium paid m times a year whose annual premium would be equal to $B_{x:k]}$ (m = 2, 3, 4, 6, 12).

If commutation numbers on monthly level are already available, it is possible to define the premium for m months by using annuities calculated for m months. The formulas are the same as in annual case except that instead of annual values, monthly values are used.

Sometimes the premium paid m times a year is based on annual commutation numbers. This approximation formula will be discussed in chapter 13.5.

However, quite often the premium paid m times a year is defined as follows:

$$B^{(m)}_{x:k]} = \frac{B_{x:k]}}{m} \cdot (1 + \eta_m)$$

where
η_m = loading coefficient related to the premium in case the premium is paid m times a year

Turning this around: The annual premium from the monthly premium is

$$B_{x:k]} = \frac{m}{1 + \eta_m} \cdot B^{(m)}_{x:k]}.$$

[32] This model could be justified if the sum insured does not increase during the insurance period, which means that also γ-loading does not increase but the costs will increase.

The approximate formula for η-loading is $\eta_m = \frac{m-1}{2 \cdot m} \cdot i$.[33] However, often more prudent loading formulas are used.

It is possible to argument both that β-loading should be charged before charging η-loading and that η-loading should be charged before charging β-loading. I have used the latter alternative.[34]

In this case the annual components are as follows:
- annual premium: $B_{x:k]}$
- annual η-loading $-\eta_m \cdot B_{x:k]} = -\frac{\eta_m}{1+\eta_m} \cdot m \cdot B^{(m)}_{x:k]}$

The η-loading is charged because the undertaking looses some interest yield because not all of the money comes already in the beginning of the year. Taken this into account, I shall consider in chapter 13.5 how η-loading should be taken into account in monthly calculations.

[33] See Neill (3.5.5) p. 88
[34] See also Koskinen et al. chapters 1.2 and 1.3 p. 102 where similar approach has been taken. This describes the performance analysis used in Finland.

9 EXAMPLES FROM FINNISH INDUSTRY PRACTICE

9.1 Loadings related to risk premiums[35]

In Finland the industry practice has been that term life is calculated as follows:

$$B_{x:k]} = \frac{(1+\varphi)\cdot\dfrac{\overline{M}_{x+t}-\overline{M}_{w}}{D_{x+t}}+\varepsilon\cdot\ddot{a}_{x:w}}{(1-\kappa)\cdot\overline{a}}\cdot S_x$$ [36]

If we replace here ε with γ and κ with β, we find the Gerber's general expense-loaded model described above with additional φ-loading.

The amount of φ-loading is found by multiplying the result given in chapter 5.4. by φ as follows:

$$\varphi\cdot\left[V_{x+t}+\frac{i}{1-q_{x+t}}\cdot V_{x+t}-\frac{q_{x+t}}{1-q_{x+t}}\cdot(S_{x+t}-V_{x+t})\right]$$

9.2 Loadings related to payments

In Finland the payment related loadings for continuous payments are κ-loading as described in chapter 9.1 and η-loading as described in chapter 8.3.

In addition to this the continuous premium is divided by 1,025.[37] This is approximately $\sqrt{1{,}045}$ (=1,0223) which means that the interest is charged for half of the year. This is not charged from single payments. My proposal is that this is taken into account in the η-loading. However, η-loading is not charged from annual payments. So, I would call the component a "payment type correction". See also chapter 13.5 below.

[35] Formulas have been revised after first edition of the document.

[36] See Vakuutusmatematikon tutkintolautakunta, Yksilöllisen henkivakuutuksen laskukaavat 2.2. - I have also seen in Estonia a technical bases where $\varphi=\sqrt{1+i}$. Because the mortality in this case was discrete, $\sqrt{1+i}$ could be interpreted as correction of mortality to the middle of the year and not as loading component (compare with continuation correction of mortality in chapter 6.4.4).

[37] See Vakuutusmatematikon tutkintolautakunta, Yksilöllisen henkivakuutuksen laskukaavat 1.6 (representing interest rate 4,5 %).

9.3 Some sickness insurance policies in Finland

As described in chapter 5.3, in Finland the risk functions are continuous and force of mortality[38] is defined by Makeham model. In this case D_x has a derivative equal to $-D_x \cdot (\delta + \mu_x)$ where $\delta = \ln(1+i)$ and μ_x is the force of mortality.[39]

On the other hand some sickness and disability covers have been defined by the formula

$$A_{x:w}(*) = (1+\omega) \cdot \left[a \cdot \ddot{a}_{x:w} + b \cdot 10^{-3} \cdot \int_x^w (0{,}1 \cdot u)^5 \cdot \frac{D_u}{D_x} du \right],$$

where ω is the safety margin and a and b are product- and gender-specific parameters.[40] So ω is essential part of the risk function and is not considered as a separate component.

By using Trapezoidal rule[41] for the integration part we obtain:

$$\int_x^w (0{,}1 \cdot u)^5 \cdot \frac{D_u}{D_x} du = \frac{1}{D_x} \cdot \int_x^w (0{,}1 \cdot u)^5 \cdot D_u du$$

$$= \frac{1}{D_x} \sum_{u=x}^{w} \left[(0{,}1 \cdot u)^5 \cdot D_u \right] - \frac{1}{2 \cdot D_x} \cdot \left[(0{,}1 \cdot x)^5 \cdot D_x + (0{,}1 \cdot w)^5 \cdot D_w \right]$$

$$+ \frac{1}{12 \cdot D_x} \cdot \left[0{,}5 \cdot (0{,}1 \cdot x)^4 \cdot D_x - (0{,}1 \cdot x)^5 \cdot D_x \cdot (\delta + \mu_x) - 0{,}5 \cdot (0{,}1 \cdot w)^4 \cdot D_w + (0{,}1 \cdot w)^5 \cdot D_w \cdot (\delta + \mu_w) \right]$$

Let us first analyze this term by term and denote the terms by V^a_{x+t}, V^b_{x+t} and V^c_{x+t} respectively.

For $V^a_{x+t+1} - \frac{D_{x+t}}{D_{x+t+1}} \cdot V^a_{x+t}$ we obtain:

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[38] See about force of mortality e.g. Gerber, pp. 16 – 17, pp. 40 – 46.
[39] See Neill p. 191 and Pesonen et al. p. 55.
[40] See Vakuutusmatemaatikon tutkintolautakunta, Yksilöllisen henkivakuutuksen laskukaavat 1.12. Previously the power in the integral was 4 instead of 5 but the method shown here may be applied also in this case (see SHY, Yksilöllisen henkivakuutuksen vakuutusmaksujen laskukaavat 1.1.2).
[41] See about Trapezoidal rule e.g. Kreuszig, pp. 869 – 872.

$$V^a_{x+t+1}-\frac{D_{x+t}}{D_{x+t+1}}\cdot V^a_{x+t}=\frac{1}{D_{x+t+1}}\cdot\sum_{u=x+t+1}^{w}\left[(0{,}1\cdot u)^5\cdot D_u\right]-\frac{D_{x+t}}{D_{x+t+1}}\cdot\frac{1}{D_{x+t}}\cdot\sum_{u=x+t}^{w}\left[(0{,}1\cdot u)^5\cdot D_u\right]$$

$$=\frac{0{,}1^5}{D_{x+t+1}}\cdot\sum_{u=x+t}^{w}\left[u^5\cdot D_u\right]-\frac{0{,}1^5\cdot D_{x+t}}{D_{x+t+1}}\cdot(x+t)^5-\frac{D_{x+t}}{D_{x+t+1}}\cdot\frac{0{,}1^5}{D_{x+t}}\cdot\sum_{u=x+t}^{w}\left[u^5\cdot D_u\right]=-\frac{D_{x+t}}{D_{x+t+1}}\cdot 0{,}1^5\cdot(x+t)^5$$

For V^b_{x+t} we obtain:

$$V^b_{x+t}=-\frac{1}{2\cdot D_{x+t}}\cdot\left[(0{,}1\cdot(x+t))^5\cdot D_{x+t}+(0{,}1\cdot w)^5\cdot D_w\right]=-\frac{0{,}1^5}{2}\cdot\left[(x+t)^5+\frac{D_w}{D_{x+t}}\cdot w^5\right]$$

and

$$V^b_{x+t+1}-\frac{D_{x+t}}{D_{x+t+1}}\cdot V^b_{x+t}=$$

$$=-\frac{0{,}1^5}{2}\cdot\left[(x+t+1)^5+\frac{D_w}{D_{x+t+1}}\cdot w^5\right]+\frac{D_{x+t}}{D_{x+t+1}}\cdot\frac{0{,}1^5}{2}\cdot\left[(x+t)^5+\frac{D_w}{D_{x+t}}\cdot w^5\right]$$

$$=-\frac{0{,}1^5}{2}\cdot(x+t+1)^5+\frac{D_{x+t}}{D_{x+t+1}}\cdot\frac{0{,}1^5}{2}\cdot(x+t)^5$$

For V^c_{x+t} we obtain:

$$V^c_{x+t}=\frac{1}{12\cdot D_x}\cdot\left[0{,}5\cdot(0{,}1\cdot x)^4\cdot D_x-(0{,}1\cdot x)^5\cdot D_x\cdot(\delta+\mu_x)-0{,}5\cdot(0{,}1\cdot w)^4\cdot D_w+(0{,}1\cdot w)^5\cdot D_w\cdot(\delta+\mu_w)\right]$$

$$=\frac{(0{,}1\cdot x)^4}{12}\cdot\left[0{,}5-0{,}1\cdot x\cdot(\delta+\mu_x)\right]-\frac{D_w}{D_x}\cdot\frac{(0{,}1\cdot w)^4}{12}\cdot\left[0{,}5-0{,}1\cdot w\cdot(\delta+\mu_w)\right]$$

and

$$V^c_{x+t+1}-\frac{D_{x+t}}{D_{x+t+1}}\cdot V^c_{x+t}=$$

$$=\frac{[0{,}1\cdot(x+t+1)]^4}{12}\cdot\left[0{,}5-0{,}1\cdot(x+t+1)\cdot(\delta+\mu_{x+t+1})\right]-\frac{D_w}{D_{x+t+1}}\cdot\frac{(0{,}1\cdot w)^4}{12}\cdot\left[0{,}5-0{,}1\cdot w\cdot(\delta+\mu_w)\right]$$

$$-\frac{D_{x+t}}{D_{x+t+1}}\cdot\frac{[0{,}1\cdot(x+t)]^4}{12}\cdot\left[0{,}5-0{,}1\cdot(x+t)\cdot(\delta+\mu_{x+t})\right]+\frac{D_{x+t}}{D_{x+t+1}}\cdot\frac{D_w}{D_{x+t}}\cdot\frac{(0{,}1\cdot w)^4}{12}\cdot\left[0{,}5-0{,}1\cdot w\cdot(\delta+\mu_w)\right]$$

$$=\frac{[0{,}1\cdot(x+t+1)]^4}{12}\cdot\left[0{,}5-0{,}1\cdot(x+t+1)\cdot(\delta+\mu_{x+t+1})\right]-\frac{D_{x+t}}{D_{x+t+1}}\cdot\frac{[0{,}1\cdot(x+t)]^4}{12}\cdot\left[0{,}5-0{,}1\cdot(x+t)\cdot(\delta+\mu_{x+t})\right]$$

Now we can see that the change of the reserve for a single premium will be as follows (assuming that $s_{x+t}=1$):

$$V_{x+t+1} - \frac{D_{x+t}}{D_{x+t+1}} \cdot V_{x+t} = a \cdot (1+\omega) \cdot \ddot{a}_{x+t+1:w} - \frac{D_{x+t}}{D_{x+t+1}} \cdot a \cdot (1+\omega) \cdot \ddot{a}_{x+t:w}$$

$$(1+\omega) \cdot \frac{b}{1000} \cdot (V^{a}_{x+t+1} - \frac{D_{x+t}}{D_{x+t+1}} \cdot V^{a}_{x+t} + V^{b}_{x+t+1} - \frac{D_{x+t}}{D_{x+t+1}} \cdot V^{b}_{x+t} + V^{c}_{x+t+1} - \frac{D_{x+t}}{D_{x+t+1}} \cdot V^{c}_{x+t})$$

$$= a \cdot (1+\omega) \cdot \frac{D_{x+t}}{D_{x+t+1}} - (1+\omega) \cdot \frac{b}{1000} \cdot \frac{D_{x+t}}{D_{x+t+1}} \cdot 0{,}1^5 \cdot (x+t)^5$$

$$+ (1+\omega) \cdot \frac{b}{1000} \cdot \left[-\frac{0{,}1^5}{2} \cdot (x+t+1)^5 + \frac{D_{x+t}}{D_{x+t+1}} \cdot \frac{0{,}1^5}{2} \cdot (x+t)^5 \right]$$

$$+ (1+\omega) \cdot \frac{b}{1000} \cdot \frac{[0{,}1 \cdot (x+t+1)]^4}{12} \cdot [0{,}5 - 0{,}1 \cdot (x+t+1) \cdot (\delta + \mu_{x+t+1})]$$

$$- (1+\omega) \cdot \frac{b}{1000} \cdot \frac{[0{,}1 \cdot (x+t)]^4}{12} \cdot \frac{D_{x+t}}{D_{x+t+1}} \cdot [0{,}5 - 0{,}1 \cdot (x+t) \cdot (\delta + \mu_{x+t})]$$

$$= a \cdot (1+\omega) \cdot \frac{1+i}{1-q_{x+t}} - (1+\omega) \cdot b \cdot \frac{1+i}{1-q_{x+t}} \cdot \frac{0{,}1^8}{2} \cdot (x+t)^5 - (1+\omega) \cdot b \cdot \frac{0{,}1^8}{2} \cdot (x+t+1)^5$$

$$+ (1+\omega) \cdot b \cdot \frac{0{,}1^7 \cdot (x+t+1)^4}{12} \cdot [0{,}5 - 0{,}1 \cdot (x+t+1) \cdot (\delta + \mu_{x+t+1})]$$

$$- (1+\omega) \cdot b \cdot \frac{0{,}1^7 \cdot (x+t)^4}{12} \cdot \frac{1+i}{1-q_{x+t}} \cdot [0{,}5 - 0{,}1 \cdot (x+t) \cdot (\delta + \mu_{x+t})]$$

$$= (1+\omega) \cdot \frac{1+i}{1-q_{x+t}} \cdot \left(a - b \cdot \frac{0{,}1^7}{4} \cdot (x+t)^4 \cdot \left\{ \frac{1}{5} \cdot (x+t) + \frac{1}{3} \cdot [0{,}5 - 0{,}1 \cdot (x+t) \cdot (\delta + \mu_{x+t})] \right\} \right)$$

$$+ (1+\omega) \cdot b \cdot \frac{0{,}1^7}{4} \cdot (x+t+1)^4 \cdot \left\{ -\frac{1}{5} \cdot (x+t+1) + \frac{1}{3} \cdot [0{,}5 - 0{,}1 \cdot (x+t+1) \cdot (\delta + \mu_{x+t+1})] \right\}$$

This shows that even for quite complicated functions it is possible to find solutions. The calculation of the change is actually much faster than calculating the net single premium because it does not have a sum over the policy period.

Discounted value in the beginning of the year is

$$= (1+\omega) \cdot \left(a - b \cdot \frac{0{,}1^7}{4} \cdot (x+t)^4 \cdot \left\{ \frac{1}{5} \cdot (x+t) + \frac{1}{3} \cdot [0{,}5 - 0{,}1 \cdot (x+t) \cdot (\delta + \mu_{x+t})] \right\} \right)$$

$$+ (1+\omega) \cdot b \cdot \frac{0{,}1^7}{4} \cdot (x+t+1)^4 \cdot \frac{1-q_{x+t}}{1+i} \cdot \left\{ -\frac{1}{5} \cdot (x+t+1) + \frac{1}{3} \cdot [0{,}5 - 0{,}1 \cdot (x+t+1) \cdot (\delta + \mu_{x+t+1})] \right\}$$

10 RISK PREMIUMS WITHOUT EFFECTS ON RESERVES

There may be covers that do not have any effects on reserves annually. For example in Finnish industry practice, waiver of premium has been such a product. In this case, waiver of premiums should be treated as payment-related loading and charged each time the payment arrives.

11 SUM AND PREMIUM CHANGES

11.1 General

In the conventional products, the level premium in the beginning of a policy period is the premium defined above.

In universal life products the loading structure can be more complicated and the only way to find a level premium can be to simulate the reserve changes and the level premium is found by iteration. In iteration the sum insured is given. Different methods can be used.[42]

When the policy is changed, then the new sum insured and premium should be found so that the savings accrued does not change at the point as the change takes place.

Above we have assumed that the formulas are valid during the whole insurance or payment period. Actually it is sufficient that they are valid through the entire calculation period (which can be shorter than a month). So, the calculation period should be divided into periods where the parameters are the same. When the value is changed at the end of the period, the initial value of the period should be used.

11.2 Sum insured

From the reserve formula we obtain the result that new sum S_{x+t}^{new} defined by other changed parameters t, V_{x+t} and $B_{x+t:k-t]}^{new}$ is

$$S_{x+t}^{new} = \frac{V_{x+t} + B_{x+t:k-t]}^{new} \cdot (1-\beta) \cdot \ddot{a}_{x+t:n-t]}}{A_{x+t:w}(*) + \gamma \cdot \ddot{a}_{x+t:n-t]}}$$

11.3 Premium

From the reserve formula we obtain the result that the new premium $B_{x+t:k-t]}^{new}$ defined by other changed parameters t, V_{x+t} and S_{x+t}^{new} is

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42 See e.g. Kreyszig, pp. 838 – 848.

$$B^{new}_{x+t:k-t]} = \frac{S^{new}_{x+t} \cdot (A_{x+t:w}(*) + \gamma \cdot \ddot{a}_{x+t:n-t]}) - V^{A}_{x+t}}{(1-\beta) \cdot \ddot{a}_{x+t:n-t]}}$$

11.4 Products with varying sum insured

The above-described method can be used also for any product where the change of sum insured has been defined beforehand and a level premium is charged. Traditionally, products with decreasing sum insured have been sold.

11.5 A case when the condition that the accrued savings are preserved is not fulfilled

11.5.1 General remarks

I described above that normally the savings accrued does not change at the point where the change takes place. I have, however, seen the general expense-loaded premium model as described in 7.1 so that α_1-loading is charged also each time the payment is increased.

$$B_{x:k]} = \frac{A_{x:w}(*) + \alpha_1 + \gamma \cdot \ddot{a}_{x:w}}{(1-\beta) \cdot \overline{a}} \cdot S_x$$

11.5.2 New sum insured

Let's define a change formula for such a case. The new sum consists from the sum insured of the old premium added by the sum insured of the payment increase (where the α_1-loading has been charged):

$$S^{new} = \frac{V_{x+t} + B_{x+t-1:k-t+1]} \cdot (1-\beta) \cdot \ddot{a}_{x+t:k-t]}}{A_{x+t:w} + \gamma \cdot \ddot{a}_{x+t:w}}$$
$$+ \frac{(B_{x+t:k-t]} - B_{x+t-1:k-t+1]}) \cdot (1-\beta) \cdot \ddot{a}_{x+t:k-t]}}{A_{x+t:w} + \gamma \cdot \ddot{a}_{x+t:w} + \alpha_1}$$

$$= \frac{V_{x+t} + B_{x+t-1:k-t+1]} \cdot (1-\beta) \cdot \ddot{a}_{x+t:k-t]}}{A_{x+t:w} + \gamma \cdot \ddot{a}_{x+t:w}}$$

$$+ \frac{(B_{x+t:k-t]} - B_{x+t-1:k-t+1]}) \cdot (1-\beta) \cdot \ddot{a}_{x+t:n-t]} \cdot \frac{A_{x+t:w} + \gamma \cdot \ddot{a}_{x+t:w}}{A_{x+t:w} + \gamma \cdot \ddot{a}_{x+t:w} + \alpha_1}}{A_{x+t:w} + \gamma \cdot \ddot{a}_{x+t:w}}$$

$$= \frac{V_{x+t} + B_{x+t:k-t]} \cdot (1-\beta) \cdot \ddot{a}_{x+t:k-t]}}{A_{x+t:w} + \gamma \cdot \ddot{a}_{x+t:w}}$$

$$- \frac{(B_{x+t:k-t]} - B_{x+t-1:k-t+1]}) \cdot (1-\beta) \cdot \ddot{a}_{x+t:k-t]}}{A_{x+t:w} + \gamma \cdot \ddot{a}_{x+t:w}}$$

$$+ \frac{(B_{x+t:k-t]} - B_{x+t-1:k-t+1]}) \cdot (1-\beta) \cdot \ddot{a}_{x+t:k-t]} \cdot \frac{A_{x+t:w} + \gamma \cdot \ddot{a}_{x+t:w}}{A_{x+t:w} + \gamma \cdot \ddot{a}_{x+t:w} + \alpha_1}}{A_{x+t:w} + \gamma \cdot \ddot{a}_{x+t:w}}$$

$$= \frac{V_{x+t} + B_{x+t:k-t]} \cdot (1-\beta) \cdot \ddot{a}_{x+t::k-t]}}{A_{x+t:w} + \gamma \cdot \ddot{a}_{x+t:w}}$$

$$- \frac{(B_{x+t:k-t]} - B_{x+t-1:k-t+1]}) \cdot (1-\beta) \cdot \ddot{a}_{x+t:k-t]} \cdot \left(1 - \frac{A_{x+t:k-t]} + \gamma \cdot \ddot{a}_{x+t:w}}{A_{x+t:k-t]} + \gamma \cdot \ddot{a}_{x+t:w} + \alpha_1}\right)}{A_{x+t:w} + \gamma \cdot \ddot{a}_{x+t:w}}$$

$$= \frac{V_{x+t} + B_{x+t:k-t]} \cdot (1-\beta) \cdot \ddot{a}_{x+t:k-t]}}{A_{x+t:w} + \gamma \cdot \ddot{a}_{x+t:w}}$$

$$- \frac{(B_{x+t:k-t]} - B_{x+t-1:k-t+1]}) \cdot (1-\beta) \cdot \ddot{a}_{x+t:k-t]} \cdot \frac{\alpha_1}{A_{x+t:w} + \gamma \cdot \ddot{a}_{x+t:w} + \alpha_1}}{A_{x+t:w} + \gamma \cdot \ddot{a}_{x+t:w}}$$

$$= \frac{V_{x+t} + \left(B_{x+t:k-t]} - \frac{\alpha_1}{A_{x+t:w} + \gamma \cdot \ddot{a}_{x+t:w} + \alpha_1} \cdot (B_{x+t:k-t]} - B_{x+t-1:kn-t+1]})\right) \cdot (1-\beta) \cdot \ddot{a}_{x+t:k-t]}}{A_{x+t:w} + \gamma \cdot \ddot{a}_{x+t:w}}$$

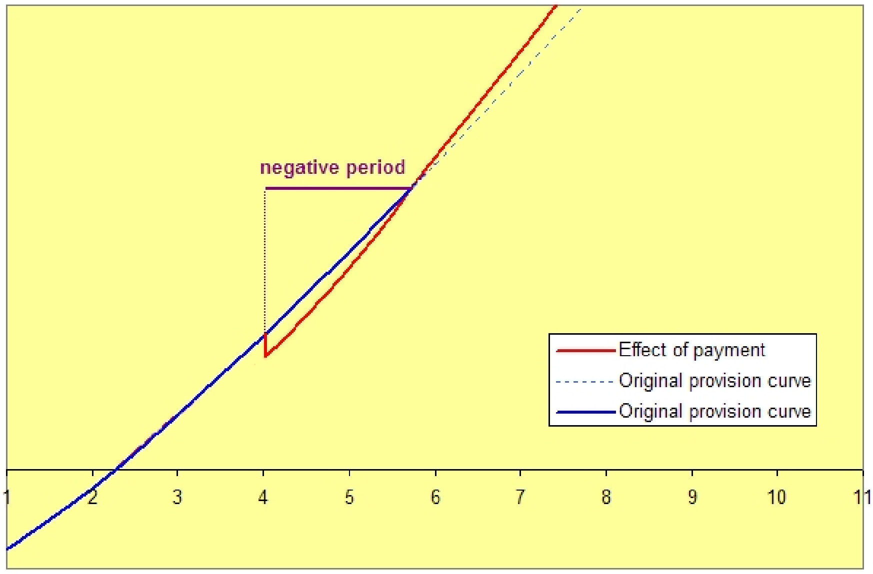
The final result is quite obvious. The charge of the α_1-loading is taken from the payment increase.

The new sum insured is calculated each time the payment is increased.

11.5.3 Reserve

In the universal life-formulas the annual α_1-loading of the change of sum insured is taken into account by charging annually $\alpha_1 \cdot (S_{x+t} - S_{x+t-1})$.

In order to preserve for the policyholder the earnings that the policyholder has earned, the negative reserve of each sum insured increase should be changed to 0. This situation is illustrated in picture 11.1.



Picture 11.1. Effect of α_1-loading charge when payment is increased related to the special case described in chapter 11

In this example the original curve is followed as long as the reserve where the effect of payment has been taken into account is bigger than the original reserve curve. It may happen that there are several such negative periods valid at the same time.

If α_1-loading had been charged, the reserve would be too low. This is the case when

$$V_{x+t} = A_{x+t:w} + \gamma \cdot \ddot{a}_{x+t:w} - \frac{A_{x:w} + \alpha_1 + \gamma \cdot \ddot{a}_{x:w}}{(1-\beta) \cdot \ddot{a}_{x:k]}} \cdot (1-\beta) \cdot \ddot{a}_{x+t:k-t]} < 0$$

which is equal to the following equations:

$$V_{x+t} = \frac{(A_{x+t:w} + \gamma \cdot \ddot{a}_{x+t:w}) \cdot \ddot{a}_{x:k]} - (A_{x:w} + \alpha_1 + \gamma \cdot \ddot{a}_{x:w}) \cdot \ddot{a}_{x+t:k-t]}}{\ddot{a}_{x:k]}} < 0,$$

$$(A_{x+t:w} + \gamma \cdot \ddot{a}'_{x+t:w}) \cdot \ddot{a}_{x:k]} - (A_{x:w} + \alpha_1 + \gamma \cdot \ddot{a}_{x:w}) \cdot \ddot{a}_{x+t:k-t]} < 0,$$

$$\frac{A_{x:w} + \alpha_1 + \gamma \cdot \ddot{a}_{x:w}}{\ddot{a}_{x:k]}} > \frac{A_{x+t:w} + \gamma \cdot \ddot{a}_{x+t:w}}{\ddot{a}_{x+t:k-t]}}$$

and

$$\frac{A_{x:w} + \gamma \cdot \ddot{a}_{x:w}}{\ddot{a}_{x:k]}} + \frac{\alpha_1}{\ddot{a}_{x:k]}} > \frac{A_{x+t:w} + \gamma \cdot \ddot{a}'_{x+t:w}}{\ddot{a}_{x+t:k-t]}}$$

This result is quite obvious: The reserve is negative as long as the future payments without α_1-loading are smaller than the payments with loading in the beginning of the period. Let us call this period "negative period".

11.5.4 Solutions

11.5.4.1 Solution of the conventional model

In the conventional model it is possible to define several cover elements, each including
- the sum insured (first the sum insured and then the sum increase)
- the payment (first the initial premium and then the premium increase)
- the cover period for the cover
- the payment period for the cover

Using conventional techniques the reserve for each cover may be calculated quite easily.

11.5.4.2 Solutions of universal life model

In the universal life –model, calculating a reserve for each cover element as we do for the conventional model would result in a great deal of data, because each payment increase creates a new account for the policy and several account entry changes would be stored into the database for each account. Also creating summaries for the customer would be quite complex.

Therefore the goal should be to find a model where the reserve is managed in one account only.

Below, I shall describe two models:
- an approximation model.
- exact model

11.5.4.3 Approximation model

In this model if $t \geq 1$, the additional reserve at moment x+t is corrected by

$$\sum_{u \geq 1}^{n_\alpha} \left\{ \left(B_{x+t+u:n-t-u]} - B_{x+t+u-1:n-t-u+1]} \right) \cdot \alpha_1 \right\}$$

where $n_\alpha = \min(t, n_\alpha^{\max})$ and $n_\alpha^{\max}$ depends on the years to maturity (n-t) and is analyzed by testing the length of the negative period as described in chapter 11.5.3.

I have made those tests on one portfolio and obtained the following results:

n-t	$n_\alpha^{\max}$
1-19	1
20-31	2
32-38	3
39-47	4
48-49	5

In practice this model means that the charged α_1-loading increments are added to the savings from those years defined above.

The reserve increments are too large but are not bigger than what was charged from the customer as a whole.

11.5.4.4 Exact model

Let us denote Δ_t by

$$\Delta_t = \frac{A_{x+t:w} + \gamma \cdot \ddot{a}_{x+t:w}}{\ddot{a}_{x+t:k-t]}} - \frac{A_{x:w} + \gamma \cdot \ddot{a}_{x:w}}{\ddot{a}_{x:k]}} - \frac{\alpha_1}{\ddot{a}_{x:k]}},$$

In this model the additional reserve at moment x+t is corrected by

$$\sum_{u}^{\substack{\Delta_u < 0 \\ 1 \leq u \leq t}} \left\{ -\Delta_u \cdot \left[S_{x+u:n-u]} - S_{x+u-1:n-u+1]} \right]^+ \cdot \ddot{a}_{x+u:k-u]} \right\}$$

This model gives exact result on an annual basis but requires some additional coding.

If the customer decides not to continue to let the payments to be increased, then it is still possible that the effects of previous payment increases should be taken into account. The issue of how old payment increases should be taken into account may be evaluated from above mentioned approximation model. The negative period depends on the time to maturity and is always less or equal than five years in the analyzed portfolio. Therefore there is no need to analyze the older changes.

11.5.5 Universal life model components

Also in the negative period the calculated α_1-loading should be shown.

The correction component could be called an α_1-compensation and it is the difference between current and previous correction terms. Because this is a correction term, it will not be added in the beginning of the year but at the end of the year.

11.5.6 Summary of the case

In this case I showed clearly that methods used in the conventional model should not be always followed in universal life model. In such a case the model would have greatly increased complexity and database size.

We could see though that by using a different approach it was still possible to find an exact fit to the old model.

It was also notable that though risk, loading and premium components could be charged in the beginning of the calculation period, some components that correct the calculated reserve should always be charged at the end of the year.

12 ZILLMERIZATION

Zillmerization for acquisition costs is normally a given percentage of the premium.[43] In the universal life model zillmerization has to be calculated at the end of each year and the change is taken into account as a zillmerization component.

If the zillmerization percentage is equal to z and the zillmerization time is m years and the level premium is $B_{x:k]}$, then each year the zillmerization change is equal to

$$Z_{x+t+1} - Z_{x+t} = -\frac{z}{m} \cdot B_{x:k]} \,.$$

Here Z_{x+t} is the zillmerization at age x+t. When the correction takes place, the zillmerization still left should be decreased from the reserve.

If the zillmerization depends on the current payment and the payment changes from time to time, then the effect is equal to

$$Z_{x+t+1} - Z_{x+t} = z \cdot \left(\frac{m-t-1}{m} \cdot B_{x+t+1:k-t-1]} - \frac{m-t}{m} \cdot B_{x+t:k-t]} \right)$$

[43] See e.g. Gerber, pp. 106 – 107.

13 CALCULATION AT THE END OF AN INSURANCE MONTH

13.1 General

The methods defined above give exact values for the end of an insurance year.

It is common that the insurance undertaking has defined the reserve formulas of the conventional products at least for each insurance anniversary. It is almost as common that some approximation formula is used between the insurance anniversaries.

In this chapter 13 I shall concentrate on defining the exact values of the reserves. Mortality will be adjusted so that the reserve of the pure endowment will be preserved. With this mortality assumption and common mortality charges, the reserve is no longer preserved. Therefore I shall define different mortality functions for such cases.

13.2 Mortality assumption at non-integer ages[44]

13.2.1 General

It is common that the mortality tables are defined for integer ages. In continuous models the mortalities for non-integer ages can be easily calculated.

The mortality in non-integer ages has been defined in literature in different ways. The most common models are the following[45]:

- uniform distribution of deaths (called also UDD or linearity of mortality)
- constant force of mortality
- Balducci model (called also hyperbolic model)

I shall refer the above-mentioned models, but I also propose some modifications to them.

Jones and Mereu have criticized the models: "While this has the advantage of simplicity, all three assumptions result in force of mortality and probability density functions with implausible discontinuities at integer ages."[46]

[44] "Non-integer ages" has been used e.g. by Forfar (see Forfar, p. 1007). Sometimes this is also called "fractional ages". (See e.g. Bowers et al. p. 74 and Jones et al.)
[45] See Jones et al. family, p. 261 – 276, Jones et al. critique, p. 363 – 370, where the authors unify and extend the mentioned models and see Bowers et al. pp. 74 – 76 Gerber p. 21 – 22 and Forfar p. 1007.
[46] See Jones et al. critique, p. 363.

My point of view is the conversion. In conversion the insurance undertaking is bound to the promises it has given. I am not concerned about eventual discontinuities. I introduce here a new concept called "discount factor preserving method" and derive some mortality functions based on that concept. The proposed modifications that I mentioned above are based on this method.

13.2.2 Discount factor preserving method

One possible goal for the universal life model is that each year the reserve is exactly the same as if it were calculated by the conventional formulas. This means that accumulation and discount factors should be the same on an annual basis:

$$\frac{D_x}{D_{x+1}} = \prod_{m=0}^{11} \frac{D_{x+m/12}}{D_{x+(m+1)/12}} = \frac{1+i}{1-q_x}$$

I shall later call this as "discount factor preserving method".

If we assume that the interest rate is constant, then in accordance with the annual accumulation factor, the monthly accumulation factor is as follows:

$$\frac{D_{x+m/12}}{D_{x+(m+1)/12}} = \frac{\sqrt[12]{1+i}}{1-q_{x+m/12}}$$

So, the monthly interest rate is equal to $\sqrt[12]{1+i}-1$.

In principle it is possible to find a discount factor preserving method by adjusting the mortality, the interest rate or both. In practice I propose to adjust mortality because the interest rate has normally been fixed.

Note that if interest is constant, then, because $D_x = l_x \cdot \left(\frac{1}{1+i}\right)^x$, discount factor preserving method preserves also l_x-numbers at the end of the year.[47]

13.2.3 Constant force of mortality

Let us denote constant force of mortality by μ^c and the respective mortality by $q^c_{x+m/12}$.

In this case l_{x+1} is equal to

[47] Jones and Mereu write about the model that I have called linear D_x-model: "Strictly speaking, this is not an FAA... (fractional age assumption) ... because different age at death distributions arise for differenct choices of the interest rate." (See Jones et al. family, p. 262.) In discount factor preserving method the mortality may depend on the chosen interest rate, but normally not vice versa. There are some arguments against the discount factor preserving models that I shall consider in summary section.

$$l_{x+1} = l_x \cdot e^{-\mu_x}$$
and
$$\mu_x = -\ln\left(\frac{l_{x+1}}{l_x}\right)$$

Accordingly $l_{x+m/12}$ is equal to

$$l_{x+(m+1)/12} = l_{x+m/12} \cdot e^{-\mu^c/12}$$

and

$$\prod_{m=0}^{11} \frac{D_{x+m/12}}{D_{x+(m+1)/12}} = \prod_{m=0}^{11} e^{\mu^c/12} = e^{\sum_{m=0}^{12} \mu^c/12} = e^{\mu^c} = e^{\mu_x} = \frac{D_x}{D_{x+1}}.$$

Hence, the mortality is

$$q^c_{x+m/12} = 1 - e^{-\mu^c/12}$$

This means that the same force of mortality for non-integer years may be used as for the integer years. In fact constant force of mortality implies also that the mortality is constant in non-integer years. Let us denote the constant mortality by q^c_x. Its value depends on q_x and may be found as follows:

$$\prod_{m=0}^{11} \frac{D_{x+m/12}}{D_{x+(m+1)/12}} = \frac{\left(\sqrt[12]{1+i}\right)^{12}}{\left(1-q^c_x\right)^{12}} = \frac{1+i}{1-q_x} = \frac{D_x}{D_{x+1}}$$

So, we obtain

$$\left(1-q^c_x\right)^{12} = 1-q_x .$$

This yields the following result:

$$q^c_x = 1 - \sqrt[12]{1-q_x}$$

So, it is possible to choose whether to use constant mortality or force of mortality.

13.2.4 Uniform distribution of deaths

The unified mortality means that the deaths are uniformly distributed. In the UDD model the l_x-numbers are interpolated as follows:

$$l_{x+m/12} = \left(1-\frac{m}{12}\right) \cdot l_x + \frac{m}{12} \cdot l_{x+1}$$

When dividing by l_x, the following result is obtained:

$$\frac{l_{x+m/12}}{l_x} = 1 - \frac{m}{12} + \frac{m}{12} \cdot \frac{l_{x+1}}{l_x} = 1 - \frac{m}{12} \cdot \left(1 - \frac{l_{x+1}}{l_x}\right) = 1 - \frac{m}{12} \cdot q_x .$$

From this we obtain

$$\frac{l_{x+(m+1)/12}}{l_{x+m/12}} = \frac{1 - \frac{m+1}{12} \cdot q_x}{1 - \frac{m}{12} \cdot q_x} = 1 - \frac{\frac{1}{12} \cdot q_x}{1 - \frac{m}{12} \cdot q_x} = 1 - \frac{q_x}{12 - m \cdot q_x}$$

which yields the result

$$q_{x+m/12} = \frac{q_x}{12 - m \cdot q_x}$$

However, this method does not preserve the discount factor. So, I shall define a modified UDD as such a mortality q_x^u that the mortality in month x+m is equal to $\frac{q_x^u}{12 - m \cdot q_x^u}$ and preserves the discount factor. Then we obtain the following equation:

$$\prod_{m=0}^{11} \left(1 - \frac{q_x^u}{12 - m \cdot q_x^u}\right) = 1 - q_x$$

Here q_x^u -number can be found by iteration.[48]

13.2.5 Balducci assumption

The Balducci assumption[49] assumes that the monthly mortality is determined by

$$\frac{1}{l_{x+m/12}} = \frac{1 - \frac{m}{12}}{l_x} + \frac{\frac{m}{12}}{l_{x+1}}$$

Because of this it is sometimes called hyperbolic model.

In this case we obtain

[48] See about iteration e.g. Kreyszig, pp. 838 – 848.
[49] See Jones et al. family, p. 261 – 276, Jones et al. critique, p. 363 – 370, Gerber p. 21 – 22 and Forfar p. 1007.

$$\frac{l_{x+1}}{l_{x+m/12}} = \frac{l_{x+1}\cdot\left(1-\frac{m}{12}\right)+l_x\cdot\frac{m}{12}}{l_x} = \frac{l_{x+1}+\frac{m}{12}\cdot\left(l_x-l_{x+1}\right)}{l_x} = \frac{l_x-\left(l_x-l_{x+1}\right)+\frac{m}{12}\cdot\left(l_x-l_{x+1}\right)}{l_x}$$

$$= \frac{l_x-\left(1-\frac{m}{12}\right)\cdot\left(l_x-l_{x+1}\right)}{l_x} = 1-\left(1-\frac{m}{12}\right)\cdot q_x$$

which is the Balducci assumption for one month.

From this we obtain

$$\frac{l_{x+(m+1)/12}}{l_{x+m/12}} = \frac{1-\left(1-\frac{m+1}{12}\right)\cdot q_x}{1-\left(1-\frac{m}{12}\right)\cdot q_x} = 1-\frac{\frac{1}{12}\cdot q_x}{1-\left(1-\frac{m}{12}\right)\cdot q_x} = 1-\frac{q_x}{12-\left(12-m\right)\cdot q_x}$$

which yields to the result

$$q_{x+m/12} = \frac{q_x}{12-\left(12-m\right)\cdot q_x}$$

This mortality does not, however, preserve the discount factor.

Let us now define modified Balducci assumption as such a mortality q_x^b that the mortality in month x+m is equal to $\frac{q_x^b}{12-(12-m)\cdot q_x^b}$ and preserves the discount factor. Then we obtain the following equation:

$$\prod_{m=0}^{11}\left[1-\left(1-\frac{m}{12}\right)\cdot q_x^b\right] = 1-q_x$$

In this case q_x^b-number can be found by iteration. [50]

13.2.6 Continuous model

In the continuous model case the monthly mortalities may be calculated from the mortality function. The same formulas as on annual level may be applied for the calculations at the end of an insurance month.

In this case l_x is defined by using continuous force of mortality:

[50] See about iteration e.g. Kreyszig, pp. 838 – 848.

$$l_x = l_0 \cdot e^{-\int\limits_0^x \mu_s ds}$$

From this we obtain also a value for each month:

$$l_{x+m/12} = l_0 \cdot e^{-\int\limits_0^{x+m/12} \mu_s ds}$$

In this case

$$D_{x+m/12} = l_0 \cdot (1+i)^{-x+m/12} \cdot e^{-\int\limits_0^{x+m/12} \mu_s ds}$$

and the accumulation factor is

$$\frac{D_{x+m/12}}{D_{x+(m+1)/12}} = \frac{l_0 \cdot (1+i)^{-x-m/12} \cdot e^{-\int\limits_0^{x+m/12} \mu_s ds}}{l_0 \cdot (1+i)^{-x-(m+1)/12} \cdot e^{-\int\limits_0^{x+(m+1)/12} \mu_s ds}} = (1+i)^{-\frac{1}{12}} \cdot e^{-\int\limits_{x+m/12}^{x+(m+1)/12} \mu_s ds}$$

Thus we obtain the desired result:

$$\prod_{m=0}^{11} \frac{D_{x+m/12}}{D_{x+(m+1)/12}} = \prod_{m=0}^{11} l_0 \cdot (1+i)^{-1/12} \cdot e^{-\int\limits_{x+m/12}^{x+(m+1)/12} \mu_s ds}$$

$$= l_0 \cdot (1+i)^{-12/12} \cdot e^{-\sum\limits_{m=0}^{11} \int\limits_{x+m/12}^{x+(m+1)/12} \mu_s ds} = l_0 \cdot (1+i)^{-1} \cdot e^{-\int\limits_x^{x+1} \mu_s ds} = \frac{D_x}{D_{x+1}}$$

From this we may calculate the q_x-numbers as in annual case.

$$q_{x+m} = 1 - \frac{l_{x+(m+1)/12}}{l_{x+m/12}} = 1 - e^{-\int\limits_{x+m/12}^{x+(m+1)/12} \mu_s ds}$$

13.2.7 Linear D_x-model

Let us assume that D_x -numbers change linearly across non-integer years.[51] This means that for all m = 0,…,11

$$D_{x+(m+1)/12} = D_{x+m/12} - \frac{1}{12} \cdot (D_x - D_{x+1}) = D_x - \frac{m+1}{12} \cdot (D_x - D_{x+1}) = \left(1 - \frac{m+1}{12}\right) \cdot D_x + \frac{m+1}{12} \cdot D_{x+1}$$

$$= \left(\frac{11-m}{12} + \frac{1-q_x}{1+i} \cdot \frac{m+1}{12}\right) \cdot D_x$$

[51] See also the comment of Jones and Mereu mentioned in the footnote of chapter 13.4.3.

On the one hand,

$$\frac{D_{x+m/12}}{D_{x+(m+1)/12}} = \frac{\sqrt[12]{1+i}}{1-q_{x+m/12}}$$

On the other hand,

$$\frac{D_{x+m/12}}{D_{x+(m+1)/12}} = \frac{\left(\frac{12-m}{12}+\frac{1-q_x}{1+i}\cdot\frac{m}{12}\right)\cdot D_x}{\left(\frac{11-m}{12}+\frac{1-q_x}{1+i}\cdot\frac{m+1}{12}\right)\cdot D_x} = \frac{(12-m)\cdot(1+i)+(1-q_x)\cdot m}{(11-m)\cdot(1+i)+(1-q_x)\cdot(m+1)}$$

From this we obtain

$$q_{x+m/12} = 1-\frac{(12-m)\cdot(1+i)+(1-q_x)\cdot m}{(11-m)\cdot(1+i)+(1-q_x)\cdot(m+1)}\cdot\sqrt[12]{1+i}$$

13.2.8 Linear discount factor model

Let us assume that the reserve of pure endowment changes linearly across non-integer years. This means that the monthly change is equal to

$$\frac{1}{12}\cdot\left(\frac{D_x}{D_{x+1}}-1\right) = \frac{1}{12}\cdot\left(\frac{1+i}{1-q_x}-1\right) = \frac{1}{12}\cdot\frac{i+q_x}{1-q_x}$$

and

$$D_{x+m/12} = \frac{D_x}{1+\frac{m}{12}\cdot\frac{i+q_x}{1-q_x}} = 12\cdot\frac{1-q_x}{12-12\cdot q_x+m\cdot(i+q_x)}\cdot D_x = 12\cdot\frac{1-q_x}{12+m\cdot i-(12-m)\cdot q_x}\cdot D_x$$

Now for all m = 0,…,11

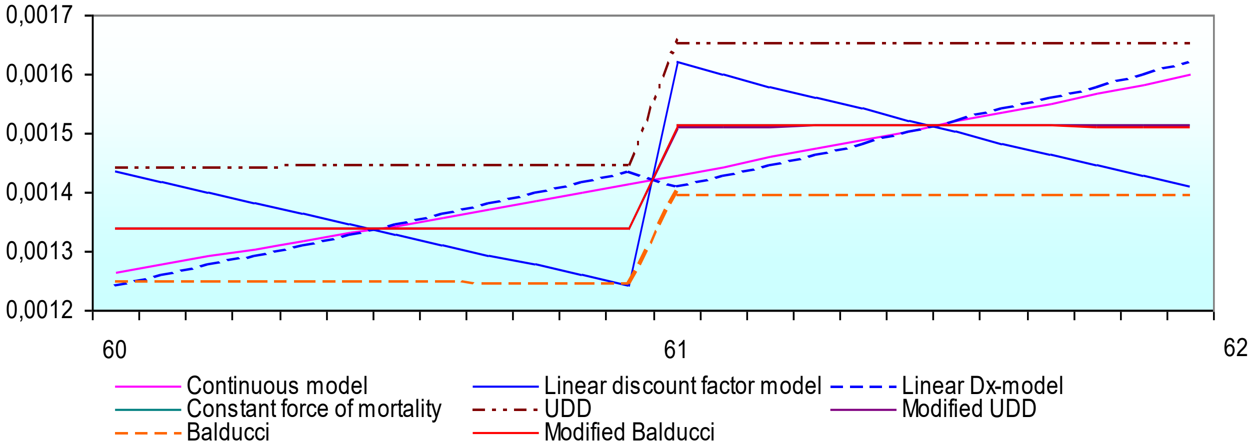
$$\frac{D_{x+m/12}}{D_{x+(m+1)/12}} = \frac{\sqrt[12]{1+i}}{1-q_{x+m/12}} = \frac{12\cdot\frac{1-q_x}{12+m\cdot i-(12-m)\cdot q_x}\cdot D_x}{12\cdot\frac{1-q_x}{12+(m+1)\cdot i-(12-(m+1))\cdot q_x}\cdot D_x} = \frac{12+(m+1)\cdot i-(11-m)\cdot q_x}{12+m\cdot i-(12-m)\cdot q_x}$$

From this equation we obtain the following result:

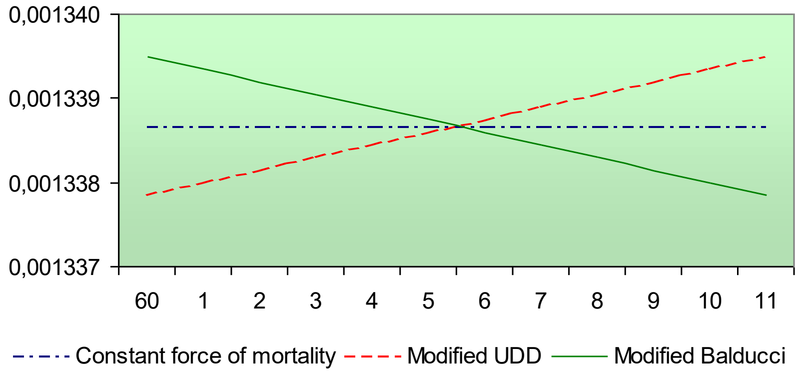
$$q_{x+m/12} = 1-\frac{12+m\cdot i-(12-m)\cdot q_x}{12+(m+1)\cdot i-(11-m)\cdot q_x}\cdot\sqrt[12]{1+i}$$

13.2.9 Summary

Above I have defined mortalities for several models. Traditional Balducci and UDD models are not discount factor preserving, but the others are. In picture 13.1 the mortalities for a man between 60 and 62 years using the Finnish force of mortality and interest rate 3,5 % are shown.[52] The scale is such that the deviations between constant force of mortality, modified UDD and modified Balducci models are not easily seen in picture 13.1 but are clear in picture 13.2.



Picture 13.1. Monthly mortalities with different mortality assumptions (deviations between constant force of mortality, UDD and modified Balducci models may not be seen from this picture, but from picture 13.2)[53]



Picture 13.2. Monthly mortalities with different mortality assumptions for constant force of mortality, modified UDD and modified Balducci models

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[52] $\mu_{x+h} = 1{,}15 \cdot (0{,}00048 \cdot + 10^{0{,}055\cdot(x+h-94{,}5)-0{,}02\cdot(x+h-72)^{+}})$ - See Yksilöllisen henkivakuutuksen laskuperusteet SHV-tutkintoa varten 1.3.1 and Pesonen et al., p. 47.

[53] I have showed here the mortality curves instead of forces of mortality curves because the peak in the shift of years and scaling would have caused that the differences of the models would have not been clearly visible.

In conversion I propose to use one of the discount preserving models. In accordance with this it can be seen from picture 13.1 that the models that give smallest and largest values, i.e. the UDD and Balducci models, should not be used.

The Balducci model has been sometimes criticized because the mortality is decreasing.[54] So also the linear discount factor model, as shown in picture 13.1. This is for the reason that the shorter interest rate accumulation period is compensated for by lower mortality.

Also Jones and Mereu criticize the models: "In specifying the FAA for each age, we wish to achieve a well-behaved force of mortality over all ages that is consistent with the life table being used."[55]

As I mentioned, my point of view is the conversion. It should be considered what is cost-effective and what are the other goals. When choosing the model we may take into account the following factors:
1) What universal life models does the undertaking support currently? – If the other products support e.g. UDD or Balducci model, then it is cost-effective to choose similar model also for the converted products.
2) How large is the portfolio that should be converted? – For small portfolios it is not cost-effective to create new customized mortality models.
3) What are the future plans related to the portfolio? – If the plan were to offer the possibility to change the policy from non-flexible to flexible policy, then a model that best suits for the flexible model would be preferred.
4) Should the universal file formulas match exactly the conventional formulas? – If e.g. the conventional formulas have linear approximation during the year, then linear discount factor preserving model should be chosen.

However, I admit that all models do not behave nicely if we look at them only from the mortality point of view. However, mortality charge is only one small element in payment and reserve structures and its importance should not be exaggerated.

13.3 Charges at non-integer ages

13.3.1 General

In previous chapter I described some mortality assumptions for discount factors. In this chapter I consider what should be charged from the policyholder or, in other words, what should be charged from the reserves. By charge in this chapter I mean risk premiums, loadings and other components charged from the reserve.

[54] See Gerber p. 22 and Forfar p. 1007.
[55] See Jones et al. critique, p. 363 – 370.

If we charge reserves monthly using the monthly interest rate and mortalities defined above, this does not preserve the reserves at the end of each insurance year. This model is possible but is an approximation that will be discussed in general in chapter 15.

The goal in this chapter is that the charge would be the same as in the policy with conventional formulas in integer years. This goal is obvious if discount factor preserving method has been used.

I consider later the following options:
1) annual charge at the end of each policy year
2) annual charge in the beginning of the insurance year
3) level premium
4) monthly charges resulting in linear reserve changes

If the charge depends on the reserve, then it should be considered what is the monthly sum insured. It is also possible to let the sum insured change due to this reason monthly, but in some cases it is reasonable not to let the sum insured change. One argument for this approach is that in the old policy the sum insured does not change monthly.

The monthly D_x- and N_x-numbers that I denote by $D_{x+t+k/12}$ and $N^{(12)}_{x+t+k/12}$ (k=0,…,12) are not the same as the annual D_x- and N_x-numbers that I denote by D_{x+t} and N_{x+t}[56]. Only in case of discount factor preserving model $D_{x+t} = D_{x+t+0/12}$. Otherwise one should limit the calculations with D_x- and N_x-numbers to one insurance year and define $D_{x+t+1} = D_{x+t+12/12}$ (actually there is one D_x- and N_x-number series for each age year).

13.3.2 Annual charge at the end of an insurance year

The annual charge can always be charged at the end of each policy year as defined in the previous chapters. This does not affect the reserve compared to the conventional methods.

However, in this case there is no charge for the ongoing insurance year in case of surrender. So, I do not recommend this option.

When deriving the other charge formulas, this is a good starting point. Let us denote this charge as P^A_{x+t+1}.[57]

[56] See also chapter 5.2.

[57] Because of this I have normally derived the charges to the end of the insurance year. In order to keep consistency with this principle I will derive also the monthly values to the end of the insurance month although in some cases the formulas would otherwise be simpler.

13.3.3 Annual charge in the beginning of an insurance year

The charge P^{A}_{x+t+1} can be discounted to the beginning of an insurance year. This option can be chosen if the argument is that the policyholder has committed to pay at least annual charges.

Let us denote this value by P^{A+}_{x+t}. In this case the value is

$$P^{A+}_{x+t} = \frac{D_{x+t+1}}{D_{x+t+0/12}} \cdot P^{A}_{x+t+1} = \prod_{m=0}^{11} \frac{D_{x+t+(m+1)/12}}{D_{x+t+m/12}} \cdot P^{A}_{x+t+1} = \frac{\prod_{m=0}^{11}(1-q_{x+t+m/12})}{1+i} \cdot P^{A}_{x+t+1}$$

If the discount factor preserving model has been used, then we may use the annual mortalities:

$$P^{A+}_{x+t} = \frac{D_{x+t+1}}{D_{x+t+0/12}} \cdot P^{A}_{x+t+1} = \frac{D_{x+t+1}}{D_{x+t}} \cdot P^{A}_{x+t+1} = \frac{1-q_{x+t}}{1+i} \cdot P^{A}_{x+t+1}$$

If the calculation period before the insurance year is not 12 but k (k=1,2,….,11) months, then use the following formula:

$$P^{A+}_{x+t} = \frac{D_{x+t+1}}{D_{x+t+k/12}} \cdot P^{A}_{x+1} = \prod_{m=12-k}^{11} \frac{D_{x+t+(m+1)/12}}{D_{x+t+m/12}} \cdot P^{A}_{x+1} = \frac{\prod_{m=12-k}^{11}(1-q_{x+t+m/12})}{1+i} \cdot P^{A}_{x+t+1}$$

At the end of the policy period the formula is

$$P^{A+}_{x+t} = \frac{D_{x+t+k/12}}{D_{x+t+0/12}} \cdot P^{A}_{x+t+1} = \prod_{m=0}^{k-1} \frac{D_{x+t+(m+1)/12}}{D_{x+t+0/12}} \cdot P^{A}_{x+1} = \frac{\prod_{m=0}^{k-1}(1-q_{x+t+m/12})}{1+i} \cdot P^{A}_{x+t+1}$$

13.3.4 Level premium

In this case the premium is charged as a level premium during the year. The monthly level premium $P_{x+m/12}$ for any m = 0,…,11 is found by dividing the annual charge P^{A}_{x+1} by the annuity.

$$P_{x+t+m/12} = \frac{D_{x+t+0/12}}{N^{(12)}_{x+t+0/12} - N^{(12)}_{x+t+12/12}} \cdot P^{A}_{x+t+1}$$

In case of discount factor preserving model this is equal to the following:

$$P_{x+t+m/12} = \frac{D_{x+t}}{N^{(12)}_{x+t+0/12} - N^{(12)}_{x+t+12/12}} \cdot P^{A}_{x+t+1}$$

Change of sum insured during the year changes also the monthly charge. The new payment is equal to

$$P_{x+t+(m+1)/12} = P_{x+t+m/12} + \frac{D_{x+t}}{N^{(12)}_{x+t+(m+1)/12} - N^{(12)}_{x+t+12/12}} \cdot (S_{x+t+1} - S_{x+t})$$

Of course, in the similar way as in annual calculations, instead of commutation numbers, summation can be used as follows:

$$\frac{N^{(12)}_{x+t+m/12} - N^{(12)}_{x+t+12/12}}{D_{x+t}} = \sum_{j=m}^{11}\left[\left(\sqrt[12]{1+i}-1\right)^{m+1} \cdot \prod_{n=1}^{m}\left(\frac{1}{1-q_{x+t+n/12}}\right)\right]$$

The same formula can be applied for any k month period (k=1,2,…,11) before the end of an insurance year. For a k month period (k=1,2,…,11 and m<k) after the end of an insurance year use the following formula:

$$\frac{N^{(12)}_{x+t+m/12} - N^{(12)}_{x+t+k/12}}{D_{x+t}} = \sum_{j=m}^{k}\left[\left(\sqrt[12]{1+i}-1\right)^{m+1} \cdot \prod_{n=1}^{m}\left(\frac{1}{1-q_{x+t+n/12}}\right)\right]$$

13.3.5 Monthly charge resulting in linear reserve changes

If the linear discount factor preserving model has been used, then it is natural to require also that the reserves change linearly during the insurance year.

Let us consider only the charge part of the reserve. The goal is to find for m = 0,…,11 a charge $P_{x+t+m/12}$ such that the monthly change of reserve is equal to ΔV.

So, the charge at the end of the first month is $P_{x+t+1/12} = \Delta V = \frac{1}{12} \cdot P^A_{x+t+1}$.

Each month the reserve of the previous month equal to $(m-1)\cdot\Delta V$ is corrected by interest rate and compensation. Thus, each month the following equation is valid:

$$P_{x+t+m/12} = \Delta V - \left(\frac{D_{x+t+(m-1)/12}}{D_{x+t+m/12}} - 1\right) \cdot (m-1) \cdot \Delta V = \left[1 - \left(\frac{D_{x+t+(m-1)/12}}{D_{x+t+m/12}} - 1\right) \cdot (m-1)\right] \cdot \Delta V$$

$$= \left(m - \frac{D_{x+t+(m-1)/12}}{D_{x+t+m/12}} \cdot (m-1)\right) \cdot \Delta V = \left(m - \frac{\sqrt[12]{1+i}}{1-q_{x+t+m/12}} \cdot (m-1)\right) \cdot \frac{1}{12} \cdot P^A_{x+t+1}$$

Especially for mortality the following is valid:

$$p_{x+t+m/12} = \frac{q_{x+t+m/12}}{1-q_{x+t+m/12}} \cdot \frac{1-q_{x+t+m/12}}{q_{x+t+m/12}} \cdot \left(m - \frac{\sqrt[12]{1+i}}{1-q_{x+t+m/12}} \cdot (m-1) \right) \cdot \frac{1}{12} \cdot P^{A}_{x+t+1}$$

$$= \frac{q_{x+t+m/12}}{1-q_{x+t+m/12}} \cdot \frac{1}{12 \cdot q_{x+t+m/12}} \cdot \left[\left(1-q_{x+t+m/12}\right) \cdot m - \sqrt[12]{1+i} \cdot (m-1) \right] \cdot P^{A}_{x+t+1}$$

This means that we may use the same monthly mortality functions if we multiply the sum insured $\frac{1}{12} \cdot P^{A}_{x+t+1}$ by:

$$\frac{1}{q_{x+t+m/12}} \cdot \left[\left(1-q_{x+t+m/12}\right) \cdot m - \sqrt[12]{1+i} \cdot (m-1) \right]$$

In table 13.1 there is an example based on the linear discount factor preserving model example presented in chapter 13.2.8. Premium is 10000, β-loading 10 %, premium multiplied by $\sqrt{1+i}$ as presented in the footnote 34 of chapter 9.1 and sum insured 50000. As a result the reserve decreases by 29,86 each month. The mortality premium decrease is first 0,30 but 0,27 at the end of the year. Total risk charge for the year is 824,29 and the premium of the first month 824,29/12 = 68,69. During the first year the risk premium decreases month by month, but later as the initial reserve is greater also the monthly premium increases.

month	premium (without β-loading)	risk premium	compensation	interest	reserve
1	9000,00	-68,69	12,95	25,88	8970,14
2	0,00	-68,40	12,74	25,79	8940,28
3	0,00	-68,10	12,54	25,70	8910,41
4	0,00	-67,81	12,33	25,62	8880,55
5	0,00	-67,53	12,13	25,53	8850,69
6	0,00	-67,24	11,93	25,44	8820,83
7	0,00	-66,96	11,74	25,36	8790,97
8	0,00	-66,68	11,54	25,27	8761,10
9	0,00	-66,40	11,35	25,18	8731,24
10	0,00	-66,12	11,16	25,10	8701,38
11	0,00	-65,85	10,98	25,01	8671,52
12	0,00	-65,58	10,79	24,93	8641,65

Table 13.1. Monthly reserve change components in case the reserve changes linearly during the year

For k month period (k=1,…,11) calibrate the annual P^{A}_{x+1} to $\frac{12}{k} \cdot P^{A}_{x+1}$.

13.4 Annuities paid several times a year

In chapter 6.3.3 I described the annual annuity model. It was shown that the annuity behaves in the same way as premiums but with negative increments. I also discussed the monthly annuities related to the payments in chapter 13.3.4.

However, quite often the monthly annuity-due is approximated by the following formulas:

$$\ddot{a}_x^{(m)} = \ddot{a}_x - \frac{m-1}{2\cdot m} \text{ and } \ddot{a}_{x:n]}^{(m)} = \ddot{a}_x - \frac{m-1}{2\cdot m}\cdot\left(1-\frac{D_{x+n}}{D_x}\right)$$ [58]

These can be written as follows:

$$\ddot{a}_x^{(m)} = \ddot{a}_x \cdot\left[1-\frac{m-1}{2\cdot m\cdot \ddot{a}_x}\right] \text{ and } \ddot{a}_{x:n]}^{(m)} = \ddot{a}_x \cdot\left[1-\frac{m-1}{2\cdot m\cdot \ddot{a}_x}\cdot\left(1-\frac{D_{x+n}}{D_x}\right)\right]$$

This means that the monthly annuity is calculated in accordance with chapter 13.3.4 taking into account the coefficient defined above.

13.5 Premiums paid several times a year

13.5.1 General

In chapter 8.3 I discussed the effects of payments if they are paid several times a year. I mentioned the case where monthly commutation numbers are available.

I mentioned also a case of η-loading. However, the loading was charged from the annual payments.

Let us now consider what is the joint effect of the payments and η-loading in monthly calculations.

In the conventional formulas, η-loading is charged to cover the interest rate and the mortality charge effect from the period before the payment. Because the η-loading coefficient is normally a constant, this always causes either a surplus or a loss. Let us assume that the η-loading structure has been defined prudently and that only a surplus is possible. (The calculations are the same but it is easier to describe the calculation results.)

[58] See Neill, p. 74 – 75. See also Bowewrs et al. p. 151 where it has been proved that in the case of linearly changed discount factors, the relations are satisfied exactly.

By $B^{(m)}_{x:k]}$ we denoted a premium paid m times a year whose annual premium would be equal to $B_{x:k]}$ (m = 2, 3, 4, 6, 12):

$$B^{(m)}_{x:k]} = \frac{B_{x:k]}}{m} \cdot (1 + \eta_m)$$

In j'th month (j = 0,…,11) payment not yet collected is equal to

$$INT\left[m \cdot \frac{12 - j}{12} \right] \cdot \frac{B_{x:k]}}{m}$$

where INT means that the number is rounded down to the nearest integer.

In order to obtain the same results as with annual payments, compensation and interest related to payments not yet collected have to be added to the reserve monthly. I call this η-correction. Some part of the payment is actual cost for the policyholder. I call this η-cost.

The total η-cost of the year is the same as the compensation and interest but calculated each month (j=1,…,12) for $INT\left[m \cdot \frac{12 - j}{12} \right] \cdot \frac{B_{x:k]}}{m}$. If we denote the monthly η-correction by ηc_j, then

$$\eta c_j = \frac{i + q_{x+j}}{1 - q_{x+j}} INT\left[m \cdot \frac{12 - j}{12} \right] \cdot \frac{B_{x:k]}}{m}$$

In cases other than monthly payments, the payments do not arrive monthly but the charges should occur monthly. In order to manage this I propose to use a ledger, Let us call it an η-ledger. From each payment the η-cost is charged immediately and the remainder is put into the η-ledger. Each month the η-correction is transferred from the ledger to the reserve.

There are several strategies for calculating the monthly η-cost related to the payment. I have devised here three alternative loading strategies:

1) Costs prioritizing method: No future η-costs are charged before the η-corrections have been either charged or transferred to the ledger.
2) Payment period method: The η-correction from months before next ordinary payment is transferred to the ledger before charging η-cost.
3) Proportional method: Each month the same annual portion of the payment is charged as η-cost.

Sometimes the premium paid m times a year is approximated by annual commutation numbers as defined in the previous chapter 13.4.[59] However, even these formulas can be transferred to the same format

$$B_{x:k]}^{(m)} = \frac{B_{x:k]}}{m} \cdot (1 + \eta_m) \, ,$$

but in this case the definition of η_m is more complicated.

The examples presented in this chapter are based on the example presented in chapter 13.3.5. The assumption in the examples is that the payment is paid four times a year. The total η-cost of the year is 172,52.

13.5.2 Costs prioritizing method

In this method no future η-costs are charged before the η-corrections have been either charged or transferred to the ledger. From table 13.2 it is possible to see that the money transferred into and out from the ledger are equal. The money into the policy is the same as monthly η-correction ηc_j.

month	premium (without β-loading)	ledger transfer		η-cost	risk premium	compensation and interest	reserve
		out	in				
1	2350,00	-100,00	29,12	0,00	-68,69	9,71	2220,14
2	0,00	0,00	29,00	0,00	-68,40	9,54	2190,28
3	0,00	0,00	28,87	0,00	-68,10	9,37	2160,41
4	2350,00	-72,52	19,17	-27,48	-67,81	18,78	4380,55
5	0,00	0,00	19,09	0,00	-67,53	18,58	4350,69
6	0,00	0,00	19,00	0,00	-67,24	18,37	4320,83
7	2350,00	0,00	9,46	-100,00	-66,96	27,63	6540,97
8	0,00	0,00	9,42	0,00	-66,68	27,39	6511,10
9	0,00	0,00	9,38	0,00	-66,40	27,15	6481,24
10	2350,00	0,00	0,00	-100,00	-66,12	36,26	8701,38
11	0,00	0,00	0,00	0,00	-65,85	35,99	8671,52
12	0,00	0,00	0,00	0,00	-65,58	35,72	8641,65

Table 13.2. Monthly reserve change components in the case of cost prioritizing method

Compared to the results presented in table 13.1 reserves match each other in months 10 – 12. Also the change of reserve is equal (-29,86) between those months where no payment takes place in the latter month.

[59] See Neill pp. 86 – 91

13.5.3 Payment period method

In this method η-correction ηc_j from months before next ordinary payment is transferred to the ledger before charging η-cost.

Compared to the example presented in table 13.2, in the payment period method the transferred money is limited to the η-correction to be collected before the next payment. This is clearly visible from table 13.3.

month	premium (without β-loading)	ledger transfer		η-cost	risk premium	compensation and interest	reserve
		out	in				
1	2350,00	-86,99	29,12	-13,01	-68,69	9,71	2220,14
2	0,00	0,00	29,00	0,00	-68,40	9,54	2190,28
3	0,00	0,00	28,87	0,00	-68,10	9,37	2160,41
4	2350,00	-57,26	19,17	-42,74	-67,81	18,78	4380,55
5	0,00	0,00	19,09	0,00	-67,53	18,58	4350,69
6	0,00	0,00	19,00	0,00	-67,24	18,37	4320,83
7	2350,00	-28,27	9,46	-71,73	-66,96	27,63	6540,97
8	0,00	0,00	9,42	0,00	-66,68	27,39	6511,10
9	0,00	0,00	9,38	0,00	-66,40	27,15	6481,24
10	2350,00	0,00	0,00	-100,00	-66,12	36,26	8701,38
11	0,00	0,00	0,00	0,00	-65,85	35,99	8671,52
12	0,00	0,00	0,00	0,00	-65,58	35,72	8641,65

Table 13.3. Monthly reserve change components in the case of payment period method

13.5.4 Proportional method

In this method each month the same annual portion of the payment is charged as the η-cost. The proportion is equal to

$$1-\frac{\sum_{j=1}^{12}\eta c_j}{B_{x:k]}\cdot\eta_m},$$

where ηc_j is the monthly η-correction ηc_j

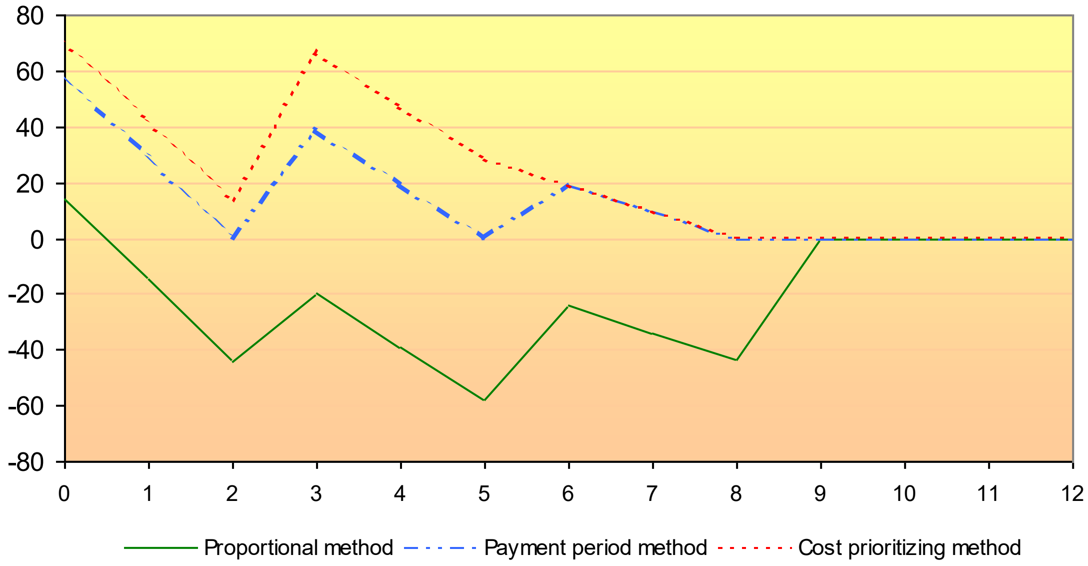
In the example shown in table 13.4 the η-cost coefficient is equal to (1-172,52/400,00 = 0,5687). So, from each payment η-cost is equal to 0,5687*100 = 56,87. From the annual payment the η-cost is 2,2 %.

month	premium (without β-loading)	ledger transfer out	in	η-cost	risk premium	compensation and interest	reserve
1	2350,00	-43,13	29,12	-56,87	-68,69	9,71	2220,14
2	0,00	0,00	29,00	0,00	-68,40	9,54	2190,28
3	0,00	0,00	28,87	0,00	-68,10	9,37	2160,41
4	2350,00	-43,13	19,17	-56,87	-67,81	18,78	4380,55
5	0,00	0,00	19,09	0,00	-67,53	18,58	4350,69
6	0,00	0,00	19,00	0,00	-67,24	18,37	4320,83
7	2350,00	-43,13	9,46	-56,87	-66,96	27,63	6540,97
8	0,00	0,00	9,42	0,00	-66,68	27,39	6511,10
9	0,00	0,00	9,38	0,00	-66,40	27,15	6481,24
10	2350,00	-43,13	0,00	-56,87	-66,12	36,26	8701,38
11	0,00	0,00	0,00	0,00	-65,85	35,99	8671,52
12	0,00	0,00	0,00	0,00	-65,58	35,72	8641,65

Table 13.4. Monthly reserve change components in the case of proportional method

13.5.5 Summary

The behavior of the ledger depends upon the chosen loading strategy, as can be seen in picture 13.3.



Picture 13.3. η-ledger in the beginning of the month and different loading strategies, as shown in tables 13.2. – 13.4

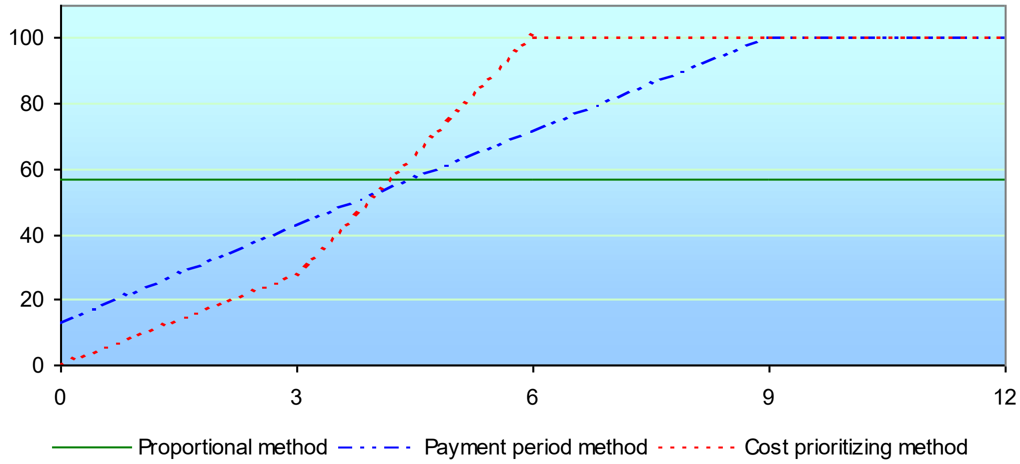
Some remarks:

1) When using the cost prioritizing and payment period methods the ledger is always non-negative. In the proportional method the ledger is sometimes negative.

2) In the payment period method the ledger is zero at the end of the payment period.
3) The ledger is zero during the last payment period.

From a bookkeeping point of view a positive ledger should be considered as prepayments and a negative ledger as unearned premium.

As seen from picture 13.4, η-costs also depend upon the chosen loading strategy



Picture 13.4. η-costs in the beginning of the month and different loading strategies, as shown in tables 13,2. – 13.4

Some remarks:
1) In the proportional method the charged η-cost does not change from time to time.
2) When using the cost prioritizing and payment period methods, the η-cost is at first lower and can be even 100 % at the end.

All strategies give the same surrender value for the policyholder. If the policyholder can see all the change components, then the proportional strategy appears to be reasonable.

However, in the proportional strategy the ledger is mostly negative. So, in case of surrender the cost charges should be corrected. From this point of view the payment period and cost prioritizing methods are better. Of these two methods the payment period method is the better because it is not possible to know when the policyholder wants to terminate the policy.

I prefer the proportional strategy, but it is possible to argument for the other strategies also.

13.6 Sum and premium changes

In chapter 13.3.1 I proposed that if the charge depends on the reserve, then the sum insured should not be allowed to change monthly.

However, it should be possible to let the sum insured be changed case by case. For example, the policyholder may ask that the policy be transferred to a paid-up policy or that the sum insured be increased.

If the discount factor preserving method has been used, then the change formulas defined in chapters 11.2 and 11.3 may be applied. The annuity-due should be calculated using the monthly chosen mortality in accordance with chapter 13.2.

However, on the day of the change there may be need for additional reserve change due to changes of the risk charge defined in chapter 13.3. Let us denote the change by C_{x+t}. In this case the formulas in chapters 11.2 and 11.3 should be replaced by the following formulas:

$$S_{x+t}^{new} = \frac{V_{x+t} + B_{x+t:k-t]}^{new} \cdot (1-\beta) \cdot \ddot{a}_{x+t:n-t]} + C_{x+t}}{A_{x+t:w}(*) + \gamma \cdot \ddot{a}_{x+t:n-t]}}$$

$$B_{x+t:k-t]}^{new} = \frac{S_{x+t}^{new} \cdot (A_{x+t:w}(*) + \gamma \cdot \ddot{a}_{x+t:n-t]}) - V_{x+t}^{A} - C_{x+t}}{(1-\beta) \cdot \ddot{a}_{x+t:n-t]}}$$

13.7 Example of monthly universal life model

In this chapter I resume the findings described above. I have chosen as an example the following endowment product:

- life tables are given for integer years
- the reserve has been linearly interpolated between the integer years
- γ-loading is charged from the sum insured and β-loading from each payment

The reserve may be calculated by

$$\begin{aligned} V_{x+t+(m+1)/12} &= V_{x+t+m/12} \\ &+ (1-\beta) \cdot B_{x+t+m/12} + \frac{\sqrt[12]{1+i} - 1}{1 - q_{x+t+m/12}} \cdot \left[V_{x+t+m/12} + (1-\beta) \cdot B_{x+t+m/12} - \gamma \cdot S_{x+t+m/12} \right] \\ &- \frac{q_{x+t+m/12}}{1 - q_{x+t+m/12}} \cdot \left[S_{x+t+m/12} - (V_{x+t+m/12} + (1-\beta) \cdot B_{x+t+m/12} - \gamma \cdot S_{x+t+m/12}) \right] \end{aligned}$$

where

$$q_{x+t+m/12} = 1 - \frac{12 + m \cdot i - (12-m) \cdot q_{x+t}}{12 + (m+1) \cdot i - (11-m) \cdot q_{x+t}} \cdot \sqrt[12]{1+i}$$

$$S_{x+t+m/12} = \frac{1}{q_{x+t+m/12}} \cdot \left[\left(1 - q_{x+t+m/12}\right) \cdot m - \sqrt[12]{1+i} \cdot (m-1)\right] \cdot \frac{q_{x+t}}{1-q_{x+t}} \cdot \frac{1}{12} \cdot S_{x+t}$$

The last term can be positive or negative depending on whether there exists a positive or a negative risk sum.

The different components are as follows:

- annual premium $B_{x+t:k-t]}$
- annual administration costs $-\gamma \cdot S_{x+t+m/12}$
- premium related costs $-\beta \cdot B_{x+t+m/12}$
- interest

$$\frac{\sqrt[12]{1+i} - 1}{1 - q_{x+t+m/12}} \cdot (V_{x+t+m/12} + (1-\beta) \cdot B_{x+t+m/12} - \gamma \cdot S_{x+t+m/12})$$

- mortality

$$-\frac{q_{x+t+m/12}}{1-q_{x+t+m/12}} \cdot \left[S_{x+t+m/12} - (V_{x+t+m/12} + (1-\beta) \cdot B_{x+t+m/12} - \gamma \cdot S_{x+t+m/12})\right]^{+}$$

- compensation

$$-\frac{q_{x+t}}{1-q_{x+t}} \cdot \left[S_{x+t+m/12} - (V_{x+t+m/12} + (1-\beta) \cdot B_{x+t+m/12} - \gamma \cdot S_{x+t+m/12})\right]^{-}$$

The values in the beginning of the month are derived in a corresponding manner.

14 CALCULATION AT THE END OF A CALENDAR MONTH

14.1 General

Because the closing date for accounting is at the end of a calendar month, there is the need to perform reserve calculation at the end of a calendar month.

There are the following options:
- Insurance month case: The calculation is performed regularly first from the end of the previous month to the insurance year date and then from that date to the end of the month.
- Calendar month case: The calculation is performed regularly only at the end of the month

14.2 Insurance month case

In the insurance month case there are at least two options:

- the value of the policy is considered to be at the end of the month the same as on the insurance month date
- there is a formula for how to calculate the value at the end of the month

Sometimes I have seen that the accuracy of the traditional formulas when calculating the reserves is one month. In that case it is natural to define that policy value is the value at the end of the month and no other calculations are needed.

One option is to use linear approximation between two monthly values:

$$V_{x+t+k/30} = V_{x+t} + k \cdot (V_{x+t+1/12} - V_{x+t})$$

where $V_{x+t+1/12}$ should be calculated without the premium if the due day of the payment coincides to the next month and $k \in [0;1]$ represents the portion of the year passed.

It is also possible to consider a general expense-loaded model with β and γ-loadings In such a case during the insurance year a possible linear approximation is as follows:

$$V_{x+t+k/30} = V_{x+t} + k \cdot (V_{x+t+1/12} - V_{x+t}) + k \cdot [(1-\beta) \cdot B - \gamma \cdot S]$$

14.3 Calendar month case

In this case the calculation is performed only at the end of the month. This means that at the same time as the conversion to the new formulas takes place the policy value is changed to the end of the month.

The new value can be found by using the approximation methods defined in chapter 15 or by some other formula, e.g. the linear approximation defined in chapter 14.2. Let us consider the effects of the linear approximation compared to the value on the insurance month date.

Let us denote the reserve based on approximation method by V^1 and based on insurance month end date by V^2 and k = 1,2,…,30 is the date of the month.

In this case we obtain the formulas for the first month after each payment:

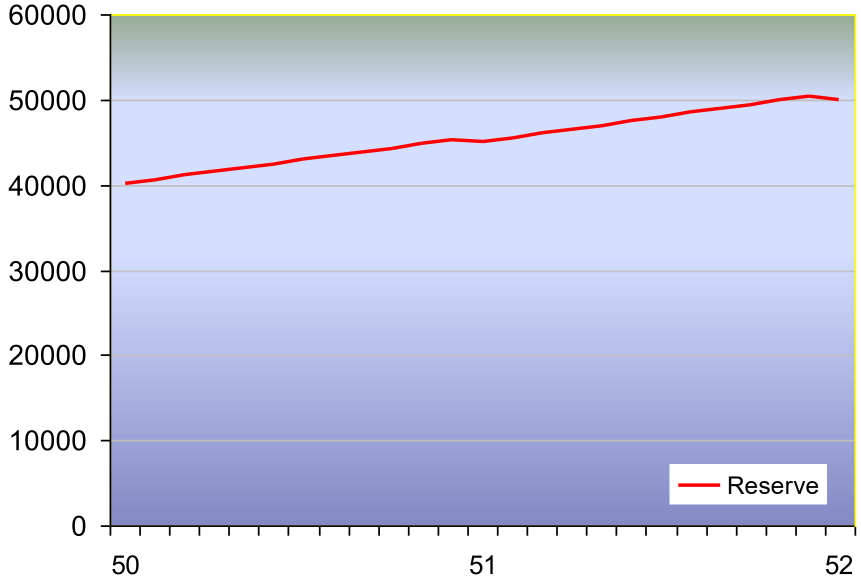
$$V^1 = V_{x+t} + k/30 \cdot (V_{x+t+1/12} - V_{x+t}) + \frac{1-k/30}{12} \cdot \left[(1-\beta) \cdot B_{x+t} - \gamma \cdot S_{x+t}\right]$$

$$V^2 = V_{x+t} + \frac{1}{12} \cdot \left[(1-\beta) \cdot B_{x+t} - \gamma \cdot S_{x+t}\right]$$

So, the difference is equal to

$$V^1 - V^2 = \frac{k/30}{12} \cdot \left[V_{x+t+1/12} - V_{x+t} - ((1-\beta) \cdot B_{x+t} - \gamma \cdot S_{x+t})\right]$$

Let us consider as an example an endowment policy for 40-year-old man with maturity age 61. In picture 14.1 the reserve has been calculated for the ages between 50 and 51 assuming that the payments are paid annually.



Picture 14.1. Correction of linear approximation at the end of an insurance year

It can be seen that the linear estimation gives in this example reserves that are larger than the actual ones, and this estimation is corrected at the end of the payment period, in this example annually.

The results are also shown in table 12.1. Each column represents the day of the month of the payment.

	Day										
Reserve	1	2	3	4	5		25	26	27	28	29	30
-100	-1,65	-3,29	-4,94	-6,58	-8,23	...	-41,14	-42,79	-44,43	-46,08	-47,73	0,00
4000	-1,38	-2,76	-4,15	-5,53	-6,91	...	-34,55	-35,94	-37,32	-38,70	-40,08	0,00
7000	-1,12	-2,25	-3,37	-4,49	-5,62	...	-28,09	-29,21	-30,33	-31,46	-32,58	0,00
11000	-0,86	-1,73	-2,59	-3,45	-4,32	...	-21,58	-22,44	-23,30	-24,16	-25,03	0,00
15000	-0,59	-1,19	-1,78	-2,38	-2,97	...	-14,86	-15,45	-16,05	-16,64	-17,24	0,00
19000	-0,31	-0,62	-0,93	-1,25	-1,56	...	-7,79	-8,10	-8,41	-8,73	-9,04	0,00
23000	0,00	0,00	0,00	0,00	-0,01	...	-0,03	-0,03	-0,03	-0,03	-0,03	0,00
27000	0,37	0,74	1,10	1,47	1,84	...	9,21	9,57	9,94	10,31	10,68	0,00
31000	0,81	1,61	2,42	3,22	4,03	...	20,15	20,96	21,77	22,57	23,38	0,00
36000	1,32	2,65	3,97	5,30	6,62	...	33,10	34,42	35,75	37,07	38,39	0,00
40000	1,89	3,77	5,66	7,54	9,43	...	47,14	49,03	50,91	52,80	54,68	0,00
45000	2,48	4,95	7,43	9,90	12,38	...	61,88	64,36	66,83	69,31	71,78	0,00
50000	3,07	6,14	9,21	12,29	15,36	...	76,79	79,86	82,93	86,01	89,08	0,00
55000	3,70	7,41	11,11	14,82	18,52	...	92,60	96,30	100,01	103,71	107,41	0,00
61000	4,40	8,80	13,21	17,61	22,01	...	110,06	114,46	118,86	123,26	127,67	0,00
66000	5,18	10,37	15,55	20,73	25,92	...	129,59	134,77	139,96	145,14	150,33	0,00
72000	6,05	12,10	18,15	24,20	30,25	...	151,23	157,28	163,33	169,38	175,43	0,00
79000	7,00	13,99	20,99	27,99	34,99	...	174,93	181,93	188,93	195,92	202,92	0,00
85000	8,06	16,11	24,17	32,23	40,28	...	201,42	209,48	217,54	225,60	233,65	0,00
93000	9,26	18,53	27,79	37,06	46,32	...	231,62	240,88	250,14	259,41	268,67	0,00
100000												

Table 12.1. Correction at the end of the month if the reserve has been linearly interpolated [60]

The average deviation for the whole portfolio (one policy for each date) is 34,56 with reserve average equal to 43715. So, the deviation is 0,079 % of the reserve. However, this is the case only in first month of the insurance year. The other months do not have deviations. So, the total deviation is 0,0066 %.

Here the chosen portfolio is from 40 to 61. If a shorter policy period, say 40 to 51, is chosen, then the values are slightly higher. In such case, deviation is 55,28, reserve average 49154, deviation 0,1124 % of the reserve and in annual level 0,0094 %. From 50 to 61 the values are 37,56, 48469, 0,077 %, 0,0065 % respectively.

[60] In order not to infringe the confidentiality of the customer, the values, except the first one, are rounded to 1000. The actual value of the first reserve is not disclosed.

In accordance with the above-mentioned example, the deviations are not large.

However, based on the above calculations it is likely that the effect is less than 1/10000 of the reserves during the first year after the conversion. Change of calculation methods does not affect after this year.

The total deviation is positive which means that the reserves should be increased.

The deviation is not final in the sense that at the end of the policy period the values are corrected. However, because of mortality and interest rate the correction is not the same as in the beginning of the policy period.

14.4 Ledger items

If the actual values are used instead of the approximation values, then it is possible that we have to user ledger as we did for several times a year paid premiums in chapter 13.5.

15 APPROXIMATION METHODS

15.1 General

In some cases it is wise to consider the possibility of changing the technical bases of the product in order to get policies that can be managed in an easier manner. This has been mentioned earlier in chapter 2.2. When deriving formulas that give exactly the same values as with conventional formulas, the models do not always look nice from actuarial point of view, especially at the monthly level, as we have already seen. The goal with the approximation methods is to find those more attractive formulas. Another goal is that the formulas belong to the same family of formulas as the other universal life products.

Here are some good candidates for the approximation:
- change from insurance month reserves to calendar month reserves
- management of several times a year paid premiums
- monthly mortality assumption

In this chapter 15 I consider the different options available for the leadership and management of the undertaking.

15.2 Client promise

I have discussed with several actuaries about the promise to the customer in case the approximation formula gives larger than actual values during a month, policy year or policy period. They have differing opinions:
- Some actuaries say that because it is mentioned in the technical bases, it is a promise to the customer and changing the calculation method to the actual one would give the policyholder the right to terminate the policy and receive the unreduced surrender value.[61]
- Some actuaries say that the promise is given only for the actual values and the actual formula can replace an approximate one, especially if the period is short (e.g. a year as in chapter 14.2).

I do not offer a final answer to this question because it depends upon the policy of the local supervisory authority. However, I would prefer the second interpretation. This option is discussed later.

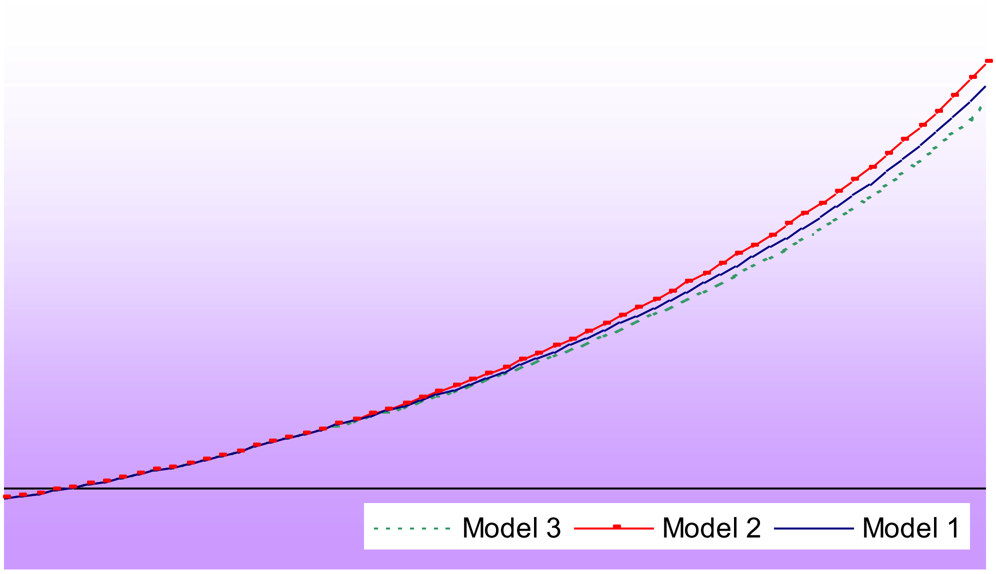
The management of the insurance undertaking should finally agree on if the value would be at least the same as the value in the old policy
- at the end of the policy period
- at the end of each insurance year
- at the end of each insurance month
- all the time

[61] See Insurance Companies Act 14:3.

15.3 Simulation

The required corrections may be analyzed at the time of the conversion by simulating the reserves policy by policy and by both old and new formulas.

The result of the simulation may be as follows (see also picture 14.1):
- The simulation gives exactly the same value (model 1). This is the optimal situation.
- The simulation with new formulas gives larger values than with old ones (model 2).
- The simulation with new formulas gives smaller values than with old ones (model 3).



Picture 14.1. Simulation results where model 1 simulates the current formulas

15.4 Increased benefits

Model 2 is for the policyholder better than the current model. From the insurance undertaking point of view the result would be negative, but in any case, the policyholder receives the same benefits that the new policyholder would.

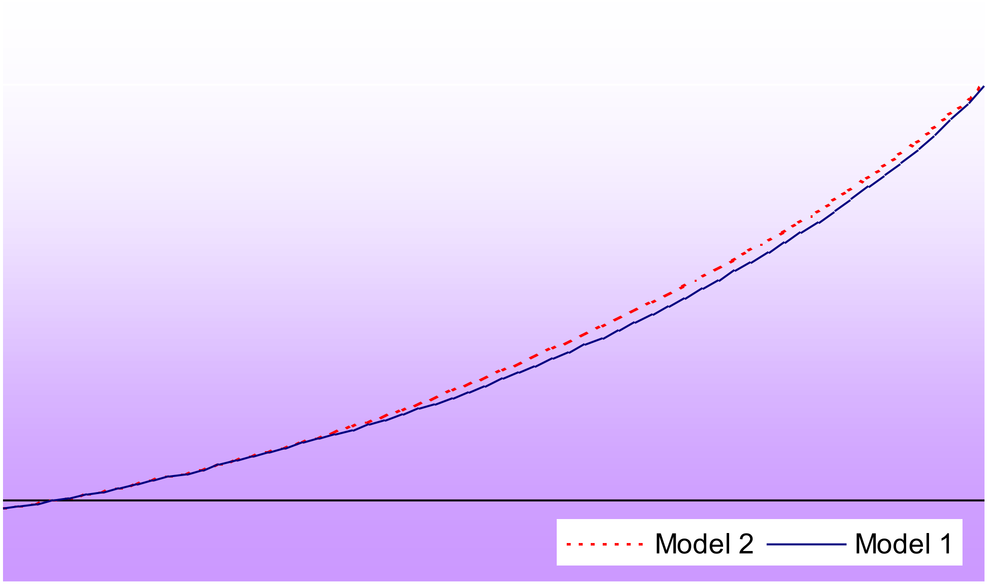
The result may be acceptable because the model 2 represents the values that the customer would get with current formulas but with old parameters. So, in a way the conditions for old and new policyholders would be equal.

15.5 Iteration

Model 3 may require that the policy value has to be adjusted so that the policyholder gets the same cover than he would get with old formulas. There are several options:

- Each time when the reserve is needed, the reserve is corrected by the change between the old and new formulas.
- At the time of the conversion the initial reserve will be increased so that the cover after the simulation would be the same as the sum insured using the old formulas (or some other criteria chosen by the management).
- Instead of adding initial reserve, some parameters of the formulas are changed. This effects during the policy period.

The desired result will be found by iteration. In iteration the sum insured is given and the required additional reserve or parameter is found by iteration.



Picture 14.2. Example of old and new reserves

After the change the progress curves of old and new reserves will be different as shown in picture 14.2.

This change can be considered to be a change of accounting policies and therefore the change is booked as changes of balance sheet in accordance with IAS 8:22 – 24 and IASB 8 and applied retrospectively.[62]

[62] See IASB 8 p. 3 – 5, 27 - 32

15.6 Rounding

Though the formulas and methods described above would give exactly the same results, in practice some companies want to show policyholders rounded numbers. This may cause small deviations, but in practice during the whole policy period the deviations should be very small. If these rounding errors are corrected, the methods described in this chapter can be considered.

16 SUMMARY

In this paper I have shown several methods for converting traditional life insurance policies into universal life policies. However, I have also proposed that common sense should be used and that it is sometimes reasonable for the insurance undertaking to expend resources for getting more unified calculation methods within the undertaking. In a long run this saves money.

This paper concentrated on the problem of finding precisely-fitting conversion formulas. When designing simpler models understanding these models is vital. In the examples related to the continuous model, modified costs during the policy period and loadings charged from premiums paid several times a year show that it is not always easy to recognize the behavior of the products. If the model is based on incorrect assumptions, then the cash flow models required in the modern insurance industry are wrong, too.

I have sometimes expressed my belief that a good actuary understands how the products behave but takes reasonable steps to simplify the model in order to get cost-effective systems. I encourage such simplifications.

The example of premium increase changes showed that sometimes it is wise to change the structure of the old cover. In the chapter where I introduced approximation methods I also called for effectiveness.

Hans U. Gerber writes about commutation numbers: "It may be … taken for granted that the days of the glory for the commutation numbers now belong to the past". His argument for this is the "advent of powerful computers" and "growing acceptance of models based on probability theory, which allows a more complete understanding of the essentials of the insurance". [63]

This is for the most part true. I still might see where in some cases use of conventional tools may be reasonable. For example, during the pension period the flexibility given by the universal life methods is not always needed. This is especially the case in statutory pension schemes. However, it is also the case that nowadays, during the pension period the investment risk is more and more often transferred to the policyholder by allowing unit-linked pensions.

In this paper I have also derived new tools to manage conversions. I have derived new concepts such as "discount factor preserving method" and methods related to premiums paid several times a year.

[63] See Gerber, p. 119.

This paper has provided tools for converting existing conventional products into universal life products. This has covered practice within the Finnish industry well. Also the discrete model has been covered and some more-difficult examples have been examined. The examples show clearly that if general actuarial principles are not followed, then the solution may be complicated but a solution may be found. Based on the guidelines given here it is straightforward to develop other formulas.

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